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RAMAN SCATTERING IN LIQUID HELIUM

AND

ROTON-ROTON INTERACTIONS

JOSEPH WOONG-SANG YAU

1971

A Dissertation Presented to the Faculty of the
Graduate School of Yale University in Candidacy for the
Degree of Doctor of Philosophy

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SUMMARY

Raman scattering in He II is found to be a second order process in the fluctuations of the dielectric susceptibility of liquid helium. The incident light polarizes a helium atom. The polarized atom interacts via dipole-dipole interaction with a second atom and induces a dipole in it, which then radiates the scattered light. This process leads to almost completely depolarized scattered light and almost isotropic distribution of the scattered light. These results are in good agreement with experiment.

The experimental spectral density of the scattered light has a single peak at a frequency shift of twice the roton energy. This can be understood if the interaction between elementary excitations is taken into account. The roton peak is shown to be due to a roton bound state of d-symmetry. The spectral distribution of the scattered light is calculated using a pseudopotential for the interaction of rotons and qualitative agreement with experiment obtained.

The pseudopotential model is inadequate to explain both roton bound states and the scattering of 2 rotons. A more realistic model is constructed for the interaction of 2 rotons using the Feynman-Cohen model of a roton. It is shown that at large separation the interaction of 2 rotons has a dipolar form which results from the interaction of the backflow fields of the rotons. At small separation the interaction is shown to be repulsive. A variational calculation of the binding energy of 2 rotons in a d-state is made using the above dipolar interaction. A bound state is obtained in reasonable agreement with experiment.

The long range interaction and bound states are shown to be unimportant in the scattering of 2 rotons. A variational calculation of the scattering of 2 rotons is made by assuming various forms for the short range repulsive interaction. Agreement with experiment is obtained if the potential is strong and has a range $1 - 2 \text{ \AA}$.

CONTENTS

	<u>Page</u>
1. Introduction	1
2. Basic Theory of Light Scattering	14
3. Raman Scattering. Second Order Process	24
4. Spectral Intensity and Roton Interactions	31
5. Roton-Roton Interactions	40
6. Roton-Roton Scattering	58
7. Conclusions	72
APPENDIX A Density Fluctuation Operator $\delta n(\vec{r}, t)$ of Liquid He II	74
APPENDIX B Evaluation of the Function $F(\vec{P}, \Omega)$	79
APPENDIX C Evaluation of Equation (6.27)	83
Acknowledgements	85
References	86
Figures	88

1. Introduction

The unique properties of superfluid helium can be explained in terms of elementary excitations which were first introduced by Landau ⁽¹⁾ in 1941. He assumed that the excitation energy spectrum has the form shown in Fig. 1. He was led to this spectrum from considerations of the thermodynamic properties of He II. Since that time the energy spectrum has been measured accurately by Henshaw and Woods ⁽²⁾ by neutron scattering. For excitations of small momentum the energy depends linearly on the momentum. These excitations are identified as thermal phonons. Excitations near the minimum energy Δ_0 are called rotons. The roton energy can be written phenomenologically as

$$E_r = \Delta_0 + (p - p_0)^2/2\mu_0$$

where $\Delta_0/k = 8.69^\circ\text{K}$, $p_0/h = 1.92 \text{ \AA}^{-1}$ and $\mu_0 = 0.16 m_{\text{He}}$ are the roton parameters. Excitations near the maximum energy Δ_1 we call 'maxons'. The maxon energy can also be written phenomenologically as

$$E_m = \Delta_1 + (p - p_1)^2/2\mu_1$$

where $\Delta_1/k = 13.94^\circ\text{K}$, $p_1/h = 1.1 \text{ \AA}^{-1}$ and $\mu_1 = 3.4 \times 10^{-24} \text{ gm}$ are the maxon parameters. We note that the roton effective mass μ_0

and maxon effective mass are of opposite sign. We will return to this point when we consider the two-particle density of states. The thermodynamic properties are determined by the rotons and phonons⁽³⁾. Below 1°K there are few rotons present because $\Delta_0 \gg kT$.

Landau's theory of the elementary excitations is essentially phenomenological. Since a superfluid is a macroscopic quantum system, we need a macroscopic wave function to describe it. However, the N-body Schrodinger equation is impossible to solve. In 1954 Feynman⁽⁴⁾ assumed, using physical arguments, that the wave function for an excited state of superfluid helium should be of the form

$$\psi^S = \sum_i f(\vec{r}_i) \phi_0 \quad (1.1)$$

where ϕ_0 is the ground state wave function and depends on the coordinates of all atoms, $f(\vec{r}_i)$ is a variational function of the radius vector of the i^{th} atom to be determined by minimizing the energy and the sum is over all atoms. The energy of the system is minimized if

$$f(r_i) = e^{i\vec{k} \cdot \vec{r}_i}$$

and the energy of the excited state is

$$\epsilon_k = \frac{k^2}{2mS(\vec{k})}$$

where $S(\vec{k})$ is the ground state liquid structure factor (from now on we set $\hbar=1$). $S(\vec{k})$ is the Fourier transform of the ground state pair correlation function. For small k , $S(\vec{k})$ is a linear function of k . This leads to the phonon spectrum. At large k , $S(\vec{k})$ is determined experimentally. It has a maximum at $k \sim 2 \text{ \AA}^{-1}$ and approaches 1 as k increases. (For large k , $S(k)$ is almost independent of temperature. The position of the maximum of $S(k)$ does not change when temperature varies.) Thus ϵ_k has a minimum at $k \sim 2 \text{ \AA}^{-1}$ with a value 19.5°K . The energy spectrum calculated from the simple Feynman wave function ψ^S is in qualitative agreement with the experimental data. However, ψ^S is not satisfactory in two respects: the predicted roton energy is too large and also the current which is associated with the localized wave packet constructed from ψ^S is not conserved. To overcome the second defect, Feynman and Cohen⁽⁵⁾ introduced a trial wave function

$$\psi = \sum_{\mathbf{i}} e^{i\mathbf{k}\cdot\vec{r}_i} e^{i \sum_{i \neq j} g(\vec{r}_j - \vec{r}_i)} \phi_0 \quad (1.2)$$

which conserves the total current. $g(r)$ describes the backflow and at large r was taken in the hydrodynamic form $A \vec{k} \cdot \vec{r} / r^3$, where A is a variational constant. The calculated roton energy using ψ has a value 11.5°K , which is a substantial improvement. The Feynman-Cohen wave function ψ provides us with a picture of a roton. We may visualize a roton as a localized excitation, a vortex ring of size comparable with the interparticle spacing. There is just enough space for an atom to pass through the center of the ring. Far from the ring there is a dipolar backflow field which is taken in hydrodynamic form. This is the picture we have in mind in our following discussion of rotons.

Experimentally, light scattering is one of the most useful tools in the study of liquid helium^(6,7,8,9). In a light scattering experiment, we measure the polarization and angular distribution, total intensity, spectral intensity and line width of the scattered light. The theory of light scattering in an isotropic homogeneous medium was first considered by Rayleigh and Einstein. The scattering is due to the thermal fluctuations of the medium. For optical frequencies, the polarizability α is independent of the frequency. The total scattered light intensity is proportional to the fourth power of the frequency. Brillouin showed that the Rayleigh line consists of a doublet due to the adiabatic density fluctuations (sound waves) and an undisplaced line due to entropy fluctuations.

In Brillouin scattering from He II, the scattered light is linearly polarized if the incident light is linearly polarized. The angular distribution of the scattered light is proportional to $\sin^2 \gamma$, where γ is the angle between the incident electric field ϵ_0 and the wave vector \vec{k} of the scattered light. In 1943 Ginzburg⁽¹⁰⁾ showed that in superfluid helium, in addition to the Brillouin doublet, the undisplaced line splits into another doublet. This is because the entropy fluctuations become propagating waves in superfluid helium (second sound). The line widths of first and second sound doublets are given by the first and second viscosities η and ξ_2 , and the thermal conductivity K , respectively. These authors obtained their results by thermodynamic and hydrodynamic considerations. A molecular theory of Rayleigh scattering from liquids was proposed by Komarov and Fisher⁽¹¹⁾ in 1962. They showed that the spectral intensity of the scattered light is proportional to the four-dimensional transform of a space-time molecular correlation function of the liquid. The importance of this theory is that we can obtain information on molecular structure and molecular motion in liquids by an inverse Fourier transform from the spectral intensity into a space-time correlation function.

We are interested in Raman scattering from He II. This work is organized in 7 sections: (1) Introduction, (2) Basic Theory of light scattering from an isotropic and homogeneous medium,

(3) Raman scattering (second order effect), (4) Spectral intensity and roton interactions, (5) Roton interactions, (6) Roton-roton scattering and (7) Conclusion.

In Section 2 we will extend Komarov and Fisher's theory to Raman scattering. We will derive the integral equation for the scattered field $\delta\vec{E}$ by considering the radiation field of an induced dipole moment. The first order effect $\delta\vec{E}^{(1)}$, which is a linear functional of the density fluctuation operator δn , describes Brillouin scattering. The second order effect $\delta\vec{E}^{(2)}$, which is a bilinear functional of δn , describes Raman scattering. As the Fourier transform $\delta n_{\vec{p}}(\omega)$ of the density fluctuation operator depends linearly on the creation operators $a_{\vec{p}}$ and annihilation operators $a_{\vec{p}}$ of excitations in the liquid, the first order light scattering involves only one excitation. By the conservation of momentum, the excitation momentum must be less than twice the momentum of the incident photon k_0 , which corresponds to the momentum of a very long wavelength phonon. Therefore Brillouin scattering of light does not involve excitations of large momentum. In particular, rotons and maxons cannot be excited in the first order process. However, it is possible to create two excitations with nearly opposite momenta \vec{p} and \vec{p}' ($\vec{p} + \vec{p}' = \vec{k} - \vec{k}_0$) in a second order process. The frequency shift of the scattered light is $\omega_p + \omega_{p'} \approx 2\omega_p$. At low temperatures only the Stokes line

of the Raman spectrum is important as the number of thermally excited rotons is small.

In Section 3 we will first obtain an expression for the Raman spectral intensity tensor $\vec{I}(\vec{r}, \omega)$. It is proportional to the Fourier transform of the 4 particle density-density correlation function $H_4 = \langle \delta n(\vec{r}_1, t_1) \delta n(\vec{r}_2, t_1) \delta n(\vec{r}_3, t_2) \delta n(\vec{r}_4, t_2) \rangle$. The polarization and angular distribution of the scattered light do not depend on the nature of the particle-particle correlation but depend only on the coupling between light and the medium. Because the scattering is a second order process, the scattered light is highly depolarized and the distribution almost isotropic. This is quite different from the case of Brillouin scattering. These predictions are in good agreement with experiment⁽⁶⁾ and give us confidence that the mechanism we have assumed for the coupling of light with helium is correct.

To determine the spectral distribution it is necessary to evaluate the two particle density of states. In Section 4 we first consider the case when there is no interaction between the excitations⁽¹²⁾. The spectral distribution of the scattered light in this case is not in good agreement with experiment. It was pointed out by Ruvalds and Zawadowski⁽¹³⁾ that the effect of interaction of the two excitations on the two-particle density of states could be important. We first assume a phenomenological separable interaction between excitations. The potential is sufficiently simple that the two-

particle density of states can be evaluated analytically. Qualitative agreement with experiment is obtained if the interaction is attractive for excitations in relative d-states. One of the most interesting features of this calculation is the prediction of a bound state of two rotons. This was not discussed in the original paper of Ruvalds and Zawadowski as they gave the rotons a very short lifetime. This bound state has been observed experimentally by Greytak et al. ⁽⁹⁾ and the measured binding energy is 0.37°K. The existence of the bound state follows automatically if the interaction is attractive. The single particle density of states for rotons is proportional to $(E - \Delta_0)^{-1/2}$ and is equivalent to that in a one-dimensional system. A bound state is then automatically obtained however weak the attractive interaction. The binding energy is proportional to the square of the interaction strength. This is similar to the case of the Cooper pairs in a superconductor except that in a metal the density of states at the Fermi surface is approximately constant and equivalent to that in a two-dimensional system. In this case a bound state is also automatically obtained for an attractive interaction. The density of scattering states is also obtained from the separable potential.

For the maxons the situation is the exact reverse of the above case. This difference arises from the fact that, in contrast to the roton having a positive effective mass μ_0 , the maxon has a negative effective mass μ_1 . If the interaction is attractive no localized

maxon state is formed. This can be simply understood as follows. If in the roton calculation the signs of the mass, the interaction and the energy are reversed the situation is unchanged. This now corresponds to the case of the maxons with a repulsive interaction. Then the maxon case with an attractive interaction is equivalent to the roton case with a repulsive interaction. In this case for the maxons we are only left with the scattering states, which do not lead to any significant features in the spectral distribution of the scattered light.

All the calculations of roton-roton interaction have been based on pseudopotentials. Ruvalds and Zawadowski⁽¹³⁾ used a potential $-g\delta(r)$ which leads to a s bound state. This state, if it exists, would not contribute to the Raman scattering. The angular distribution and polarization of the scattered light indicates that the rotons are in a relative d state. In Section 4 we then assume a potential $V(\vec{p}_1, \vec{p}_2) = -5g_2 p_2 (\cos\theta)$ for the roton interaction where θ is the angle between the incoming and outgoing momenta \vec{p}_1 and \vec{p}_2 of the rotons. For this interaction we calculate the binding energy and density of states of two rotons. The spectrum calculated in this way is in qualitative agreement with the experiments of Greytak and Yan⁽⁶⁾. The experimentally observed binding energy of two rotons leads to a value of $g_2 = 10^{-38}$ erg cm³.

Further information on roton-roton interactions is provided by the line width of the Raman spectrum⁽⁸⁾. The line width is interpreted as arising from scattering of the two rotons created by other thermally excited rotons. This leads to a value for the collision frequency of two rotons

$$\tau^{-1} = B n_r$$

where n_r is the number of thermally excited rotons and B is approximately independent of temperature and of order of magnitude $2 \times 10^{-10} \text{ cm}^3 \text{ sec}^{-1}$. The measured value⁽¹⁴⁾ of the roton viscosity of He II also provided information on the scattering of two rotons and leads to a value of the collision frequency in good agreement with the value obtained from line widths.

Landau and Khalatnikov⁽¹⁵⁾ have considered the scattering of two rotons by assuming a pseudopotential of the form $V_0 \delta(\vec{r})$. They used Born approximation and found

$$\tau_{LK}^{-1} = 2 \mu_0 p_0 V_0^2 n_r$$

where μ_0 and p_0 are roton parameters. This has the correct temperature dependence, which was their main concern. Agreement with experiment is obtained if $V_0 = 2 \times 10^{-38} \text{ erg cm}^3$ which is an order of magnitude greater than the above value of g_2 found from the bound state.

It is then impossible to understand the formation of roton bound states and the observed value of the roton collision frequencies by means of a pseudopotential. In Section 6, we show that there is an upper limit, independent of V_0 , for the scattering of two rotons by the pseudopotential of Landau and Khalatnikov. This upper limit is smaller than the experimental value by a factor of 4. It is then necessary to investigate in more detail the real interaction of two rotons. This is done in Section 5. We expect that at large separations two rotons interact via their backflows only. At a short distance their interaction is more complicated, but we expect it to be repulsive so that two rotons remain two separate elementary excitations. We investigate the long range part V_L and short range part V_S of the roton-roton interaction separately. The best description of a roton has been given by Feynman⁽⁴⁾ and Feynman and Cohen⁽⁵⁾. As was mentioned before, far from the roton core there is a dipolar backflow. We show that the interaction of the backflow fields of two rotons leads to a V_L of dipolar form. The same interaction is obtained for two small spheres moving in an ideal fluid at a separation $r \gg$ radius of the sphere from classical hydrodynamics. A third approach to the problem is to consider that two rotons interact by phonon exchange. This also leads to a dipolar interaction. Therefore we are confident that the long range part V_L of the roton interaction is a dipole-dipole interaction. This is attractive for certain orientations of the

roton momenta. However, the picture of backflow is meaningful only if the separation r is much larger than b , the size of the roton core. b can be estimated from the strength of V_L and is approximately equal to the diameter of a helium atom. Therefore, the roton-roton interaction is described by V_L only for $r \gg b$. The short range part V_S of the roton-roton interaction is intimately related to the structure of the roton core. The Feynman-Cohen picture does not provide much information about the core, though it indicates that the short range interaction V_S must be repulsive. As we have less understanding of the core structure, we will assume that V_S has a strength V_0 and a range b .

In Section 5 a variational calculation of the binding energy of two rotons in a d-state interacting via the long range dipolar interaction is made. A binding energy of approximately 0.40°K is found in reasonable agreement with experiment. The size of the bound state can be estimated from experiment and is approximately $(\mu E)^{-1/2} = 13\text{\AA}$. The variational calculation also leads to value of the radius of the two roton state in agreement with this value. Whether a bound state of s symmetry is obtained as in the case of Ruvalds and Zawadowski is somewhat doubtful. The repulsive short range potential will be important for such a state.

In Section 6 we investigate the scattering of two rotons. We first use the pseudopotential of Landau and Khalatnikov and show that there is an upper limit independent of the strength of the potential. This upper limit is too small to explain the observed scattering. The effect of bound states on the scattering is also investigated using this potential and shown to be small. The long range dipolar interaction V_L can also lead to the scattering of two rotons. This has been investigated by Toigo⁽¹⁶⁾ using the Born approximation. He showed that the collision frequency obtained from V_L is too small by a factor of 20 to explain the observed scattering. We thus conclude that the long range and attractive part of the roton-roton interaction is not important in the scattering of two rotons. The scattering must arise from the short range repulsive interaction. Our result on the Landau-Khalatnikov pseudopotential indicates that the range of the particle is also important. In Section 6 a variational calculation of the scattering of two rotons is made using several forms of repulsive potential with strength V_0 and range b . The scattering is independent of the strength of the potential if it is large enough. The range is important and with $b \sim 1 - 2\text{\AA}$ agreement with experiment is obtained. This value of b is comparable with the radius of a helium atom.

2. Basic Theory of Light Scattering

2.1 The Integral Equation of Light Propagation (17)

We consider the propagation of an electromagnetic wave in a homogeneous, isotropic non-magnetic medium. We need consider only the electric field and neglect the magnetic field as the velocity of the charges in the medium is very small compared with the light velocity. We assume that the matter is made up of molecules (or atoms in the case of helium) which are polarizable. An external field induces electric and magnetic moments in the molecules. As a first approximation it may be assumed that the polarization is a linear function of the field. The polarization vector \vec{P} can be written as

$$\vec{P}(\vec{r}, t) = n(\vec{r}, t) \alpha(\omega) \vec{E}'(\vec{r}, t) \quad (2.1)$$

where $n(\vec{r}, t)$ is the number of molecules per unit volume at \vec{r} at time t , $\alpha(\omega)$ is the polarizability and \vec{E}' is the local field. For optical frequencies, α is the electronic polarizability and varies very little with frequency. For liquid helium, α has the value $0.20 \times 10^{-24} \text{ cm}^3$.

To derive the field equation, we consider the local field \vec{E}'_j on the j^{th} atom. \vec{E}'_j is the sum of the applied field and fields of all the other dipoles in the medium:

$$\vec{E}'_j = \vec{E} + \sum_{\ell} \vec{E}_{\ell j}$$

where the prime in the summation means the sum extends over all atoms except the j^{th} atom. For an oscillating linear dipole $\vec{p}(t)$ at point \vec{r}' , the Hertz vector $\vec{\pi}$ at point \vec{r} at time t is given by

$$\vec{\pi}(\vec{r}, t) = \vec{p}(t - |\vec{r} - \vec{r}'|/c) / |\vec{r} - \vec{r}'|$$

and the field \vec{E}_d due to \vec{p} is given by

$$\vec{E}_d(\vec{r}, t) = \nabla \times (\nabla \times \vec{\pi}(\vec{r}, t)) .$$

The distribution of dipoles may be treated as continuous. Thus the local field can be written as

$$\vec{E}'(\vec{r}, t) = \vec{E}(\vec{r}, t) + \alpha \int_{\sigma} dV' \nabla \times [\nabla \times \frac{n(\vec{r}', t - R/c) \vec{E}'(\vec{r}', t - R/c)}{R}] \quad (2.2)$$

where $\vec{R} = \vec{r} - \vec{r}'$ and ∇ operates on \vec{r} only. If \vec{r} is inside the medium, the integral must exclude the volume occupied by the atom at \vec{r} . We consider it as a small sphere σ with radius a . If \vec{r} is outside the medium, the integral is taken throughout the volume V of the irradiated medium. We introduce a Green's function $\vec{g}_k(\vec{r} - \vec{r}', t - t')$ defined by

$$\vec{g}_k(\vec{r} - \vec{r}', t - t') = (\nabla \nabla - \hat{I} \nabla^2) \frac{e^{i k |\vec{r} - \vec{r}'| - i \omega(t - t')}}{|\vec{r} - \vec{r}'|} \quad (2.3)$$

where $\overleftrightarrow{\mathbf{I}}$ is a unit dyad and $k = \omega/c$. Then Eq. (2.2) can be written as

$$\vec{E}'(\vec{r}, t) = \vec{E}(\vec{r}, t) + \alpha \int_{\sigma}^V d^3r' \int_{-\infty}^{\infty} dt' \frac{d\omega'}{2} \overleftrightarrow{g}_k(\vec{r} - \vec{r}', t - t') \vec{E}'(\vec{r}', t') n(\vec{r}', t'). \quad (2.4)$$

This is the integral equation for the local field. We are interested in light scattering. The density $n(\vec{r}, t)$ can be separated into a uniform part n_0 and a fluctuating part $\delta n(\vec{r}, t)$:

$$n(\vec{r}, t) = n_0 + \delta n(\vec{r}, t) \quad (2.5)$$

where $n_0 \gg \delta n$. Let us expand \vec{E}' in a series in δn . We first introduce an expansion parameter λ and replace δn by $\lambda \delta n$ in Eq. (2.5). Then we can expand \vec{E}' in a series in λ to give

$$\vec{E}(\vec{r}, t) = \vec{E}^{(0)}(\vec{r}, t) + \lambda \vec{E}^{(1)}(\vec{r}, t) + \lambda^2 \vec{E}^{(2)}(\vec{r}, t) + \dots,$$

such that the zero, first, etc., powers of λ correspond to the zero, first, etc., orders of δn . In the final results, λ is set equal to 1. Substituting \vec{E}' and n into Eq. (2.4), we

obtain the following equations:

$$\vec{E}^{(0)}(\vec{r}, t) = \vec{E}(\vec{r}, t) + n_0 \alpha \int_{\sigma} d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}^{(0)}(\vec{r}', t'), \quad (2.6)$$

$$\begin{aligned} \vec{E}^{(1)}(\vec{r}, t) = & \alpha \int d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \cdot \vec{E}^{(1)}(\vec{r}', t') \delta n(\vec{r}', t') \\ & + n_0 \alpha \int_{\sigma} d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}^{(1)}(\vec{r}', t'), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \vec{E}^{(2)}(\vec{r}, t) = & \alpha \int d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}^{(1)}(\vec{r}', t') \delta n(\vec{r}', t') + n_0 \alpha \int_{\sigma} d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}^{(2)}(\vec{r}', t') \end{aligned} \quad (2.8)$$

and similar equations for $\vec{E}^{(3)}$, $\vec{E}^{(4)}$... etc. To understand the various terms in three equations, let us assume that the density of the medium is uniform. Then Eq. (2.4) reduces to

$$\begin{aligned} \vec{E}'(\vec{r}', t) = & \vec{E}(\vec{r}, t) + N_0 \alpha \int d^3 r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}'(\vec{r}', t') \end{aligned} \quad (2.9)$$

This equation determines the local field in a uniform medium.

If the refractive index of the medium is very nearly equal to 1, then the local field is not much different from the external field, and for all practical purposes we can neglect the second term in Eq. (2.9). For liquid helium, the refractive index is 1.028.

(The refractive index of water is 1.334.) Therefore, we can neglect the second term in Eqs. (2.6), (2.7) and (2.8). Then we obtain the following set of field equations for the scattered fields:

$$\vec{E}^{(0)}(\vec{r}, t) = \vec{E}(\vec{r}, t),$$

$$\begin{aligned} \vec{E}^{(1)}(\vec{r}, t) = & \alpha \int d^3r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}^{(0)}(\vec{r}', t') \delta n(\vec{r}', t'), \end{aligned} \quad (2.10)$$

$$\begin{aligned} \vec{E}^{(2)}(\vec{r}, t) = & \alpha \int d^3r' dt' \frac{d\omega'}{2\pi} \vec{g}_{k'}(\vec{r} - \vec{r}', t - t') \\ & \cdot \vec{E}^{(1)}(\vec{r}', t') \delta n(\vec{r}', t'), \end{aligned} \quad (2.11)$$

etc.. The scattered field $\delta\vec{E}$ is given by

$$\delta\vec{E}(\vec{r}, t) = \vec{E}^{(1)}(\vec{r}, t) + \vec{E}^{(2)}(\vec{r}, t) + \dots \quad (2.12)$$

In Eq. (2.12) we have set $\lambda = 1$.

In this work we assume that the incident field $\vec{E}(\vec{r}, t)$ is a linearly polarized monochromatic wave

$$\vec{E}(\vec{r}, t) = \vec{\epsilon}_0 e^{i\vec{k}_0 \cdot \vec{r} - i\omega_0 t}$$

where $\vec{\epsilon}_0$ is a constant vector along the direction of the polarization. The incident intensity is $I_0 = (c/8\pi)\epsilon_0^2$.

2.2 Spectral Intensity Tensor $I_{ij}(\vec{r}, \omega)$

The spectral intensity I_{ij} contains all the information relevant to a light scattering experiment. We first define the correlation function of the scattered field

$$R_{ij}(\vec{r}, \tau) = \frac{c}{8\pi} \overline{\langle \delta E_i(\vec{r}, t) \delta E_j^*(\vec{r}, t + \tau) \rangle_0}$$

where $\langle \dots \rangle_0$ indicates an average over the Gibb's ensemble of particle distributions at time t , and the bar indicates an average over a time $2T$ which is needed for the measurement of the scattered light. As T is much longer than the characteristic times of molecular processes, we may let $T \rightarrow \infty$. The correlation function $R_{ij}(\tau)$ gives the total intensity tensor of the scattered light if $\tau \rightarrow 0$. As $\tau \rightarrow \infty$, there is no correlation between the scattered fields $\delta \vec{E}(t)$ and $\delta \vec{E}(t+\tau)$, and $R_{ij}(\tau)$ approaches zero. The spectral intensity of light scattered at frequency ω

is the Fourier transform of Eq. (2.13)

$$I_{ij}(\vec{r}, \omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} R_{ij}(\vec{r}, \tau) \quad (2.14)$$

The inverse transform is

$$R_{ij}(\vec{r}, \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} I_{ij}(\vec{r}, \omega) \quad (2.15)$$

The total intensity $R_{ij}(\vec{r}, \tau = 0)$ is the integral of the spectral intensity.

The plane of polarization of the scattered light is easily obtained if we note that, in the wave zone $r \gg r'$, the Green's function $\vec{g}_k(\vec{r} - \vec{r}', t - t')$ of Eq. (2.3) becomes

$$\vec{g}_k(\vec{r} - \vec{r}', t - t') \simeq f_k(\vec{r} - \vec{r}', t - t') (\hat{I} - \hat{k} \hat{k}) \quad (2.16)$$

where

$$f_k(\vec{r} - \vec{r}', t - t') = \frac{k^2}{r} e^{ikr - i\vec{k} \cdot \vec{r}' - i\omega(t - t')}$$

with $\vec{k} = \frac{\omega}{c} \frac{\vec{r}}{r}$ and $\hat{k} = \vec{k}/k$. Thus $I_{ij}(\vec{r}, \omega)$ has non-zero components only in the plane with normal along \vec{k} . It can then be reduced to principal axes, and the ratio of its two principal values gives the degree of depolarization, while their sum is the total spectral intensity at frequency ω .

2.3 First Order Process

The first order process is described by

$$\begin{aligned}
 R_{ij}(\vec{r}, \tau) &= \frac{c}{8\pi} \overline{\langle \delta E_i^{(1)}(\vec{r}, t) \delta E_j^{(1)*}(\vec{r}, t + \tau) \rangle_0} \\
 &= I_0 \alpha^2 (\hat{i} \cdot \hat{\epsilon}_0)(\hat{j} \cdot \hat{\epsilon}_0) \int d^3r_1 dt_1 \int d^3r_2 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \\
 &\quad \overline{f_{k_1}(\vec{r} - \vec{r}_1, t - t_1) f_{k_2}(\vec{r} - \vec{r}_2, t - t_2) \langle \delta n(\vec{r}_1, t_1) \delta n(\vec{r}_2, t_2) \rangle_0} \\
 &\quad e^{i\omega_2 \tau + i\vec{k}_0 \cdot (\vec{r}_1 - \vec{r}_2) - i\omega_0(t_1 - t_2)}
 \end{aligned} \tag{2.17}$$

where \hat{i} and \hat{j} are two orthogonal unit vectors in the polarization plane. The only nonvanishing component of the intensity tensor is $R_{\epsilon_0 \epsilon_0} \equiv R_{||}$. Therefore the scattered light is linearly polarized in the first order process. The spectral intensity can be obtained by changing the variables from $(\vec{r}_1, t_1; \vec{r}_2, t_2)$ to $(\vec{r}_1, t_1; \vec{r}_1 - \vec{r}_2, t_1 - t_2)$ as the density correlation function $H_2(\vec{r}_1, t_1; \vec{r}_2, t_2) \equiv \langle \delta n(\vec{r}_1, t_1) \delta n(\vec{r}_2, t_2) \rangle_0$ is a function of relative coordinates only. Performing the remaining integral we obtain the spectral intensity

$$\begin{aligned}
 I_{||}(\vec{r}, \omega) &= \frac{I_0 V}{r^2} \frac{\alpha^2 \omega^4}{c^4} \sin^2 \gamma \int d^3r_{12} dt_{12} H_2(\vec{r}_1 - \vec{r}_2, t_1 - t_2) \\
 &\quad e^{-i(\vec{k} - \vec{k}_0) \cdot \vec{r}_{12} + i(\omega - \omega_0)t_{12}}
 \end{aligned} \tag{2.18}$$

where γ is the angle between ϵ_0 and \hat{k} , and V is the volume of the irradiated medium ($V = 1$). Then the spectral intensity,

$$I_{11}(\vec{r}, \omega) = \frac{I_0}{r^2} \alpha^2 \sin^2 \gamma \left(\frac{\omega}{c} \right)^4 H_2(\vec{K}, \Omega') \quad (2.19)$$

with $\vec{K} = \vec{k} - \vec{k}_0$ and $\Omega' = \omega - \omega_0$. It is a function of the Fourier transform $H_2(\vec{K}, \Omega')$ of the density-density correlation function $H_2(\vec{r}_{12}, t_{12})$ only.

In general the calculation of the density-density correlation function $H_2(\vec{r}, t)$ is difficult. For non-interacting excitations with small momenta $q = |\vec{k} - \vec{k}_0|$, it can be evaluated readily. From Appendix A, the Fourier transform $\delta n_{\vec{q}}(t)$ of the density fluctuation operator δn is given by

$$\delta n_{\vec{q}}(t) = \frac{1}{2i} \left(\frac{2n_0 q^2}{m\omega} \right)^{1/2} \left(a_{-\vec{q}}^+ e^{i\omega_{-\vec{q}} t} - a_{\vec{q}} e^{-i\omega_{\vec{q}} t} \right) \quad (2.20)$$

Then

$$H_2(\vec{K}, \Omega') = \frac{n_0 q^2}{2m\omega_q} \delta_{\vec{K}, \vec{q}} \left\{ \langle a_{\vec{q}}^+ a_{\vec{q}} \rangle \delta(\Omega' - \omega_q) + \langle a_{-\vec{q}} a_{-\vec{q}}^+ \rangle \delta(\Omega' + \omega_{-\vec{q}}) \right\} \quad (2.21)$$

As $q \sim 10^{-3} \text{ \AA}^{-1}$ is very small, $\langle a_{\vec{q}}^+ a_{\vec{q}} \rangle_0 \approx kT/\omega_{\vec{q}}$ and $\langle a_{-\vec{q}} a_{-\vec{q}} \rangle_0 \approx 1 + kT/\omega_{\vec{q}} \approx kT/\omega_{\vec{q}}$. (We have put $\omega_{\vec{q}} = \omega_{-\vec{q}}$ as the liquid is isotropic.) Therefore, we obtain, for noninteracting excitations,

$$I_{11}(\vec{r}, \omega) = \frac{I_0}{r^2} \frac{n_0}{m} \frac{\alpha^2}{u^2} \left(\frac{\omega}{c} \right)^4 kT \sin^2 \gamma \quad (2.22)$$

where u is the phonon velocity. The scattered light is very weak as a result of the smallness of α .

$$I_{11} \approx \left(\frac{I_0}{r^2} \right) 10^{-8} T \sin^2 \gamma .$$

In real superfluid helium, at $T > 1^\circ\text{K}$ the interactions among the excitations cannot be ignored. As a result of frequent collision, there are collective modes. In liquid helium II, the collective modes are first and second sound. Then it is better to calculate $H_2(\vec{r}, t)$ by means of hydrodynamics and thermodynamics. This situation has been investigated by Ginzburg⁽¹⁰⁾. It is found that the scattered intensity in the second sound doublet is 10^{-4} times smaller than that in first sound doublet. The line widths are determined by the first and second viscosity η and ξ_2 and thermal conductivity K . We will not pursue this matter further as our interest is in Raman scattering, which will be shown in the next section to be a second order effect in δn .

3. Raman Scattering Second Order Process

In the first order scattered field $\vec{E}^{(1)}(\vec{r}, t)$ excitations with large momentum cannot be excited because of the smallness of the photon momentum. In order to investigate rotons and maxons, the second order effect must be considered. The second order scattered field $\vec{E}^{(2)}(\vec{r}, t)$ is a bilinear functional of δn . The Raman spectral intensity is determined by the 4 particle density-density correlation function $H_4 = \langle \delta n_1 \delta n_2 \delta n_3 \delta n_4 \rangle_0$, which will be considered in more detail in Section 4. In this section we first obtain an expression for the spectral intensity in terms of the Fourier transform of the 4 particle density-density correlation function H_4 . We also show that the Raman scattered light is highly depolarized and its angular distribution is almost isotropic, as a result of the scattering being a second order process. This result is independent of any assumptions about the density-density correlation function.

3.1 Raman Spectral Intensity

The second order field $\vec{E}^{(2)}$ is given by the equation (2.11)

$$\vec{E}^{(2)}(\vec{r}, t) = \alpha^2 (\vec{I} - \hat{k} \hat{k}) \cdot \int d^4 x_1 d^4 x_2 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{d\pi} \quad (3.1)$$

$$f_{k_1}(x - x_1) \overset{\leftrightarrow}{g}_{k_2}(x_1 - x_2) \cdot \vec{E}(x_2)$$

$$\delta n(x_1) \delta n(x_2)$$

where (x_1) stands for (\vec{r}_1, t_1) , etc. As in the case of $\vec{E}^{(1)}$, the polarization of $\vec{E}^{(2)}$ is in the plane whose normal is along \hat{k} . Let \hat{p}_1 be a unit vector in this plane. Then

$$\hat{p}_1 \cdot \vec{E}^{(2)}(\vec{r}, t) = \epsilon_0 \alpha^2 \int d^4x_1 d^4x_2 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} (\hat{p}_1 \cdot \vec{g}_{k_2} (x_1 - x_2) \cdot \hat{\epsilon}_0) f_{k_1}(x - x_1) \delta n(x_1) \delta n(x_2) e^{i\vec{k}_0 \cdot \vec{r}_2 - i\omega_0 t_2} \quad (3.2)$$

The correlation function of the scattered field, including the second order effect, can be written as

$$R_{ij}(\tau) = \frac{c}{8\pi} \overline{\langle (E_i^{(1)}(t) + E_i^{(2)}(t)) (E_j^{(1)}(t+\tau) + E_j^{(2)}(t+\tau))^* \rangle_0} \quad (3.3)$$

$$= R_{ij}^{(1)}(\tau) + R_{ij}^{(2)} + \frac{c}{8\pi} \overline{\langle E_i^{(2)}(t) E_j^{(1)*}(t+\tau) + E_i^{(1)}(t) E_j^{(2)*}(t+\tau) \rangle_0}$$

where $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$ are the correlation functions of the first order and second order scattered field. The cross term is a functional of three particle correlation function $H_3 \equiv \langle \delta n(\vec{r}_1 t_1) \delta n(\vec{r}_2 t_2) \delta n(\vec{r}_3 t_3) \rangle_0$. In Appendix A, δn_q is shown to be a linear function of creation and annihilation operator of excitations, a_p^+ and a_p in the hydrodynamic approximation. Then H_3 is zero in this approximation and

$$R_{ij}(\tau) = R_{ij}^{(1)}(\tau) + R_{ij}^{(2)}(\tau) \quad (3.4)$$

Raman spectrum is determined by the Fourier transform of $R_{ij}^{(2)}(\tau)$.

From Eq. (3.1), the second order correlation function

is given by

$$R_{P_1 P_1}^{(2)}(\tau) = I_0 \alpha^4 \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \frac{d\omega_4}{2\pi} \quad (3.5)$$

$$\begin{aligned} & (\hat{P}_1 \cdot \vec{e}_{k_2}(x_1 - x_2) \cdot \hat{\epsilon}_0) (\hat{P}_1 \cdot \vec{e}_{k_4}(x_1 - x_4) \cdot \hat{\epsilon}_0) H_4(x_1, x_2, x_3, x_4) \\ & \frac{f_{k_1}(x - x_1) f_{k_3}(x - x_3)}{e^{i\omega_3 \tau} e^{i\vec{k}_0 \cdot \vec{r}_{24} - i\omega_0 t_{24}}} \end{aligned}$$

where $H_4(x_1, x_2, x_3, x_4) \equiv \langle \delta n(x_1) \delta n(x_2) \delta n(x_3) \delta n(x_4) \rangle_0$ is the 4 particle density-density correlation function. H_4 is a function of relative coordinates only. The spectral intensity can be obtained by changing variables from (x_1, x_2, x_3, x_4) into $(x_1, x_1 - x_2, x_3 - x_4, x_1 - x_3)$.

$$\begin{aligned} I_{P_1 P_1}^{(2)}(\tau) &= \frac{I_0}{r^2} \left(\frac{\alpha \omega}{c} \right)^4 \int d^3 r_{12} d^3 r_{23} d^3 r_{13} dt_{13} \\ & (\hat{P}_1 \cdot \vec{d}(\vec{r}_{12}) \cdot \hat{\epsilon}_0) (\hat{P}_1 \cdot \vec{d}(\vec{r}_{34}) \cdot \hat{\epsilon}_0) \\ & H_4(\vec{r}_{12}, \vec{r}_{24}, \vec{r}_{13}, t_{13}) e^{-i\vec{K} \cdot \vec{r}_{13} + i\Omega t_{13}} \end{aligned}$$

where $\vec{k} = \vec{k} - \vec{k}_0$ and $\Omega = \omega - \omega_0$ are the momentum and energy transfer between the incoming and outgoing photon, and

$$\vec{d}(\vec{r}_{12}) = \frac{(\vec{I} - \hat{n}_{12}\hat{n}_{12})}{r_{12}^3}.$$

Here we have made the approximation

$$\vec{g}_k(\vec{r}_{12}, t_{12}) = \vec{d} e^{-i\omega_1 t_{12}}$$

because only the short range correlations are important. The correlation function H_4 can be expanded as

$$H_4(\vec{r}_{12}, \vec{r}_{34}, \vec{r}_{13}, t_{13}) = \sum_{q_1 q_2 K'} \int \frac{d\Omega'}{2\pi} H_4(\vec{q}_1, \vec{q}_2, \vec{K}', \Omega') e^{i\vec{q}_1 \cdot \vec{r}_{12} + i\vec{q}_2 \cdot \vec{r}_{34} + \vec{K}' \cdot \vec{r}_{13} - i\Omega' t_{13}}. \quad (3.7)$$

Substituting Eq. (3.7) into $I_{P_1 P_1}^{(2)}(\omega)$ we obtain

$$I_{P_1 P_1}^{(2)}(\omega) = \frac{I_0}{r^2} \left(\frac{c\omega}{c}\right)^4 \sum_{q_1 q_2} (4\pi)^2 A(q_1)A(q_2) H_4(\vec{q}_1, \vec{q}_2, \vec{K}, \Omega') \quad (3.8)$$

$$(\hat{p}_1 \cdot (\vec{I} - 3\hat{q}_1\hat{q}_1) \cdot \hat{\epsilon}_0)(\hat{p}_1 \cdot (\vec{I} - 3\hat{q}_2\hat{q}_2) \cdot \hat{\epsilon}_0)$$

where $A(q_1) = j_1(q_1 a)/q_1 a$ with a equal to one half the average distance between two atoms in liquid helium. The momentum transfer $\vec{K} = \vec{k} - \vec{k}_0$ is very small compared with the momenta of maxons or rotons and can be neglected. $H_4(\vec{q}_1, \vec{q}_2, 0, \Omega')$ depends on Ω' , q_1 , q_2 and \vec{q}_1, \vec{q}_2 . We can now consider the polarization and angular distribution of the scattered light.

3.2 Polarization and Angular Distribution of Raman Scattered Light

To study the polarization and angular distribution of the scattered light, we expand $H_4(\vec{q}_1, \vec{q}_2, 0, \Omega')$ in spherical harmonics

$$H_4(\vec{q}_1, \vec{q}_2, 0, \Omega') = \sum_{\ell} h_{\ell}(q_1, q_2, \Omega') P_{\ell}(\hat{q}_1 \cdot \hat{q}_2).$$

Then Eq. (3.8) can be written as

$$I_{11}^{(2)}(\omega) = \sum_{\ell} \sum_{\vec{q}_1} \sum_{\vec{q}_2} R_{\ell}(q_1, q_2, \Omega') P_{\ell}(\hat{q}_1 \cdot \hat{q}_2) \quad (3.9)$$

$$(\hat{p}_1 \cdot (\hat{I} - 3\hat{q}_1\hat{q}_1) \cdot \hat{\epsilon}_0)(\hat{p}_1 \cdot (\hat{I} - 3\hat{q}_2\hat{q}_2) \cdot \hat{\epsilon}_0)$$

R_{ℓ} is a function of q_1 , q_2 and Ω' only.

Let $\hat{\epsilon}_0$ be along the z-axis and the angles of \hat{p}_1 be (θ_0, ϕ_0) . The function $\hat{p}_1 \cdot (3\hat{q}_1\hat{q}_1 - \hat{I}) \cdot \hat{\epsilon}_0$ can also be expressed in spherical

harmonics

$$\begin{aligned} \hat{P}_1 \cdot (3\hat{q}_1\hat{q}_1 - \vec{I}) \cdot \epsilon_0 &= \cos \theta_0 (3 \cos^2 \theta_1 - 1) + 3 \sin \theta_0 \cos \theta_1 \sin \theta_1 \\ &\quad \cos(\phi_0 - \phi_1) \\ &= \frac{16\pi}{5} \left(\cos \theta_0 Y_2^0 - \frac{3}{8} \sin \theta_0 (Y_2^1 e^{i\phi_0} + Y_2^{1*} e^{-i\phi_0}) \right) \end{aligned}$$

We note that only the terms with $l=2$ in H_4 contribute to $I_{P_1 P_1}^{(2)}$. In the next section we will show that $H_4(\vec{q}_1, \vec{q}_2, 0, \Omega')$ is proportional to the two-excitation density of states

$$\begin{aligned} \rho_2(\vec{q}_1, \vec{q}_2, \Omega) &\equiv \sum_n \langle 0 | a_{\vec{q}_1} a_{-\vec{q}_1} | n \rangle \langle n | a_{\vec{q}_2}^+ a_{-\vec{q}_2}^+ | 0 \rangle \\ &\quad \delta(\Omega - \omega_{no}) \quad , \end{aligned}$$

where $\Omega = -\Omega'$. Therefore, the measured two-excitation states must have d-symmetry.

The angular integrals in Eq. (3.9) can be evaluated easily.

We obtain

$$I_{P_1 P_1}^{(2)} \sim (\cos^2 \theta_0 + \frac{3}{4} \sin^2 \theta_0) \quad (3.10)$$

where θ_0 is the angle between direction of polarization of incident light $\hat{\epsilon}_0$ and direction of polarization of scattered light p_1 . The scattered light is highly depolarized. We consider two special orientations of $\hat{\epsilon}_0$ and \hat{k} , where \hat{k} is the direction of the scattered light wave vector. Let \hat{k} be along the z-axis, then p_1 must be in the xy-plane. With the notation $\omega_{i,j} = I_{ii}^{(2)}(\hat{\epsilon}_0 = \hat{j})$, where the first index of $\omega_{i,j}$ refers to the direction of \hat{p}_L and the second refers to the direction of $\hat{\epsilon}_0$, we obtain the depolarization factors $\omega_{y,x}/\omega_{x,x} = 3/4$ and $\omega_{y,z}/\omega_{x,z} = 1$. The ratio of total scattered light intensity is

$$\frac{\omega_{y,z} + \omega_{x,z}}{\omega_{y,x} + \omega_{x,x}} = \frac{6}{7}.$$

The experimental values for these ratios are 0.9 ± 0.2 , 1.15 ± 0.20 and 0.84 ± 0.08 , respectively⁽⁶⁾. The predicted values are in good agreement with the experimental values. This agreement gives us confidence that our assumptions about the coupling mechanism between the light and liquid helium are correct. The depolarization of the scattered light is independent of the nature of the particle-particle correlation.

4. Spectral Intensity and Roton Interactions

To calculate the spectral intensity, it is necessary to know the 4 particle density-density correlation function $H_4(\vec{r}_{12}, \vec{r}_{23}, \vec{r}_{13}, t_{13}) = \langle \delta n(\vec{r}_1 t_1) \delta n(\vec{r}_2 t_1) \delta n(\vec{r}_3 t_3) \delta n(\vec{r}_4 t_3) \rangle_0$. We will replace the Gibb's ensemble average $\langle \dots \rangle_0$ by a ground state average $\langle \dots \rangle$ as we are interested in excitations with energy $\gg kT$. The Fourier transform is given by $H_4(\vec{q}_1, \vec{q}_2, 0, t_{13}) = \langle \delta n_{\vec{q}_1}(\vec{r}_1) \delta n_{-\vec{q}_1}(\vec{r}_1) \delta n_{\vec{q}_2}(\vec{r}_3) \delta n_{-\vec{q}_2}(\vec{r}_3) \rangle$ where t_1 can be earlier or later than t_3 . The Fourier transform $\delta n_{\vec{q}}(t)$ of the density fluctuation operator is given in Appendix A. Substituting $\delta n_{\vec{q}}(t)$ we obtain

$$\begin{aligned}
 H_4(\vec{q}_1, \vec{q}_2, 0, \Omega') &= \pi h(q_1) h(q_2) \sum_n \{ \langle 0 | a_{\vec{q}_1} a_{-\vec{q}_1} | n \rangle \\
 &\quad \langle n | a_{\vec{q}_2}^+ a_{-\vec{q}_2}^+ | 0 \rangle \\
 &\quad + \langle 0 | a_{\vec{q}_2} a_{-\vec{q}_2} | n \rangle \langle n | a_{\vec{q}_1}^+ a_{-\vec{q}_1}^+ | 0 \rangle \} \\
 &\quad \delta(\Omega' + \omega_{n0})
 \end{aligned} \tag{4.1}$$

where $|n\rangle$ is a complete set of states and $h(q)$ is equal to $n_0 q^2 / 2\pi m \omega_q$. Since \vec{q}_1 and \vec{q}_2 are dummy variables in

$\sum_{\vec{q}_1, \vec{q}_2} H_2(\vec{q}_1, \vec{q}_2, 0, \Omega')$, we can write

$$H_4(\vec{q}_1, \vec{q}_2, 0, \Omega') = 2\pi h(q_1)h(q_2) \sum_n \langle 0 | a_{\vec{q}_1} a_{-\vec{q}_1} | n \rangle \quad (4.2)$$

$$\langle n | a_{\vec{q}_2}^+ a_{-\vec{q}_2}^+ | 0 \rangle \delta(\Omega' + \omega_{no}) .$$

To use the perturbation method, we introduce one- and two-particle Green's function

$$G_1(\vec{r}_1, \vec{r}_2; t_1, t_2) = -i \langle T(\psi(\vec{r}_1, t_1) \psi^+(\vec{r}_2, t_2)) \rangle \quad (4.3)$$

$$G_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4; t_1, t_3) = (-i)^2 \langle T(\psi(\vec{r}_1, t_1) \psi(\vec{r}_2, t_1) \psi^+(\vec{r}_3, t_3) \psi^+(\vec{r}_4, t_3)) \rangle \quad (4.4)$$

where

$$\psi(\mathbf{r}, t) = \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} a_{\vec{q}}(t) . \quad (4.4a)$$

The Fourier transforms are

$$G_1(\vec{q}, \omega) = (\omega - \omega_q + i \epsilon/2)^{-1} \quad (4.5)$$

$$G_2(\vec{q}_1, \vec{q}_2, 0, \Omega) = \frac{1}{i} \sum_n \frac{\langle 0 | a_{\vec{q}_1} a_{-\vec{q}_1} | n \rangle \langle n | a_{\vec{q}_2}^+ a_{-\vec{q}_2}^+ | 0 \rangle}{\Omega - \omega_{no} + i\epsilon} \quad (4.6)$$

where $\epsilon \rightarrow 0^+$. Then we have

$$H_4(\vec{q}_1, \vec{q}_2, 0, -\Omega) = -2h(q_1) h(q_2) \text{Re}(T_2(\vec{q}_1, \vec{q}_2, 0, \Omega)) . \quad (4.7)$$

We now calculate $G_2(\vec{q}_1, \vec{q}_2, 0, \Omega)$ assuming $G_1(\vec{q}_1, \omega)$ is known from the single particle excitation energy spectrum .

$$G_1(q, \omega) = (\omega - \epsilon_{\vec{q}} + i \epsilon/2)^{-1}$$

where $\epsilon_{\vec{q}}$ is the single particle excitation energy. The two-particle Green's function is determined by the Bethe-Salpeter equation

$$G_2 = G_2^0 + G_2^0 V G_2 \quad (4.8)$$

as we are interested in the scattering of two rotons. G_2^0 is the free two-particle Green's function. In the summation of the perturbation series, we keep the ladder diagrams only. We assume the potential is separable $V(\vec{q}_1, \vec{q}_2) = v(\vec{q}_1) v(\vec{q}_2)$. A straightforward calculation leads to

$$G_2(\vec{q}_1, \vec{q}_2, 0, \Omega) = (\delta_{\vec{q}_1, \vec{q}_2} + \delta_{\vec{q}_1, -\vec{q}_2}) F_0(\vec{q}_1, \Omega) + \frac{2i v(\vec{q}_1) v(\vec{q}_2) F_0(\vec{q}_1, \Omega) F_0(\vec{q}_2, \Omega)}{1 - F(\Omega)} \quad (4.9)$$

where

$$F_0(\vec{q}, \Omega) = \int \frac{d\omega}{2\pi} G_1(\vec{q}, \omega) G_1(-\vec{q}, \Omega - \omega) \quad (4.10)$$

and

$$F(\Omega) = i \sum_{\vec{q}} F_0(\vec{q}, \Omega) v^2(\vec{q}) . \quad (4.11)$$

We first consider the case of noninteracting excitations. (12)

Then

$$\begin{aligned} G_2(\vec{q}_1, \vec{q}_2, 0, \Omega) &= (\delta_{\vec{q}_1, \vec{q}_2} + \delta_{\vec{q}_1, -\vec{q}_2}) F_0(\vec{q}_1, \Omega) \\ &= -i (\delta_{\vec{q}_1, \vec{q}_2} + \delta_{\vec{q}_1, -\vec{q}_2}) (\Omega - 2\varepsilon_{\vec{q}} + \varepsilon)^{-1} . \end{aligned}$$

The spectral intensity for noninteracting excitations is given by

$$\begin{aligned} I_{p_1 p_1}^{(2)}(\Omega) &= 2 \frac{I_0}{r^2} \left(\frac{\alpha \omega}{c} \right)^4 (4\pi)^2 \sum_{\vec{q}_1} A^2(q_1) h^2(q_1) (\hat{p}_1 (\vec{1} - 3\hat{q}_1 \hat{q}_1) \cdot \varepsilon_0)^2 \\ &\quad \delta(\Omega - 2\varepsilon_{\vec{q}}) \end{aligned} \quad (4.12)$$

where $\Omega = \omega_0 - \omega$ is the frequency shift. The roton energy is

$\varepsilon q = \Delta_0 + (q - p_0)^2 / 2\mu_0$ and the maxon energy is $\varepsilon q = \Delta_1 - (q - p_1)^2 / 2|\mu_1|$. Therefore $I_{p_1 p_1}^{(2)}(\omega)$ has two peaks at $\Omega = 2\Delta_0$

and $\Omega = 2\Delta_1$.

$$I_{P_1 P_1}^{(2)}(\Omega) = \frac{24}{5} \frac{I_0}{r^2} \left(\frac{\alpha\omega}{c} \right)^4 [A^2(p_0) h^2(p_0) P_0^2 \frac{\mu_0}{\Omega - 2\Delta_0} \theta(\Omega - 2\Delta_0) + A_1^2(p_1) h^2(p_1) P_1^2 \frac{|\mu_1|}{2\Delta_1 - \Omega} \theta(2\Delta_1 - \Omega)] \quad (4.13)$$

where $\theta(x)$ is a step function such that $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$. The density of states is proportional to $(\Omega - 2\Delta_0)^{-1/2}$ near $\Omega = 2\Delta_0$ and $(2\Delta_1 - \Omega)^{-1/2}$ near $\Omega = 2\Delta_1$. This is similar to that of a one-dimensional system.

Experimentally only the peak near $\Omega = 2\Delta_0$ has been observed⁽⁶⁾. The predicted peak near $\Omega = 2\Delta_1$ is not present. This indicates that the interaction between excitations must be taken into account to understand the Raman spectrum of liquid He II⁽¹³⁾.

We assume a simple separable potential $V(\vec{q}_1, \vec{q}_2)$ with the d-symmetry for the interaction between excitations

$$V(\vec{q}_1, \vec{q}_2) = -5 g_2 P_2(\hat{q}_1 \hat{q}_2) \quad (4.14)$$

where g_2 is a constant and $P_2(\hat{q}_1 \hat{q}_2)$ is the Legendre polynomial of order 2. Then the function $F(\Omega)$ of Eq. (4.11) can be evaluated by considering

$$\begin{aligned} v(\vec{q}_1) F(\Omega) v(\vec{q}_3) &= i \int_{\mathbf{q}} v(\vec{q}_1) v(\vec{q}) v(\vec{q}) v(\vec{q}_3) F_0(\vec{q}, \Omega) \\ &= i \int_{\mathbf{q}} V(\vec{q}_1, \vec{q}) V(\vec{q}_3, \vec{q}) F_0(\vec{q}, \Omega) \end{aligned}$$

Then Eq. (4.9) becomes

$$G_2(\vec{q}_1, \vec{q}_2, 0, \Omega) = (\delta\vec{q}_1, \vec{q}_2 + \delta\vec{q}_1, \vec{q}_2) F_o(\vec{q}, \Omega) \quad (4.15)$$

$$= \frac{i 10 g_2 P_2(\hat{q}_1, \hat{q}_2) F_o(\vec{q}_1, \Omega) F_o(\vec{q}_2, \Omega)}{1 + i g_2 \sum_q F_o(\vec{q}, \Omega)}$$

The spectral intensity $I_{P_1 P_1}^{(2)}(\Omega)$ near $\Omega = 2\Delta_o$ is then equal to

$$I_{\epsilon_o \epsilon_o}^{(2)}(\Omega) = -J(p_o) \operatorname{Re} \left[\frac{f(\Omega)}{1 + i q_2 f(\Omega)} \right] \quad (4.16)$$

where we have taken $\hat{p}_1 \parallel \hat{\epsilon}_o$ and

$$J(p_o) = -\frac{16}{5} \frac{I_o}{r^2} \left(\frac{\alpha \omega}{c} \right)^4 (4\pi)^2 A^2(p_o) h^2(p_o)$$

$$f(\Omega) = (-i) \int_q (\Omega - 2\Delta_o - (q - p_o)^2 / 2\mu_o + i\epsilon)^{-1} \quad (4.17)$$

From Appendix B we have

$$f(\Omega) = -\frac{\mu_o p_o^2}{2\pi\sqrt{\mu_o e}} \quad \text{if } e \equiv \Omega - 2\Delta_o > 0$$

$$= i \frac{\mu_o p_o^2}{2\pi\sqrt{\mu_o b}} \quad \text{if } -b \equiv \Omega - 2\Delta_o < 0$$

We have a bound state if

$$1 + i g_2 f(\Omega) = 0 \quad (4.18)$$

The binding energy

$$b = \frac{\mu_0 p_0^4}{4\pi^2} g_2^2 = g_2^2 B^2$$

is a function of the square of g_2 . Experimentally the roton binding energy is $0.37^\circ\text{K}^{(9)}$. This value of b leads to a $g_2 = 1.10^{-39} \text{ erg cm}^3$.

The spectral intensity near $\Omega = 2\Delta_0$ is given by

$$I_{\epsilon_0 \epsilon_0}^{(2)}(\Omega)_{\text{rot}} = J(p_0) [\pi g_2^2 B^2 \delta(\Omega - 2\Delta_0 + b) + \frac{B \sqrt{e}}{e^2 + g_2^2 B^2} \theta(\Omega - 2\Delta_0)] \quad (4.19)$$

where $\theta(x)$ is a step function such that $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$. Therefore the spectral intensity near $\Omega = 2\Delta_0$ has a peak at frequency shift $\Omega = 2\Delta_0 - b$ and a continuous distribution for $\Omega > 2\Delta_0$.

The spectral density near $\Omega = 2\Delta_1$ can be calculated similarly

$$I_{\epsilon_0 \epsilon_0}^{(2)}(\Omega)_{\max} = -J(p_1) \operatorname{Re} \left[\frac{\bar{f}(\Omega)}{1 + i g_2 \bar{f}(\Omega)} \right] \quad (4.20)$$

where

$$J(p_1) = -\frac{16}{5} \frac{I_0}{r^2} \left(\frac{\alpha \omega}{c} \right)^4 (4\pi)^2 A^2(p_1) h^2(p_1)$$

and

$$\bar{f}(\Omega) = (-i) \int_q (\Omega - 2\Delta_1 + (q - p_1)^2 / |\mu_1| + i\epsilon)^{-1}.$$

$\bar{f}(\Omega)$ can be obtained from $f(\Omega)$ by changing the sign of the effective mass and is given by

$$\begin{aligned} \bar{f}(\Omega) &= -\frac{|\mu_1| p_0^2}{2\pi \sqrt{|\mu_1| \bar{e}}} \quad \text{if} \quad \bar{e} \equiv 2\Delta_1 - \Omega > 0, \\ &= -i \frac{|\mu_1| p_0^2}{2\pi \sqrt{|\mu_1| \bar{b}}} \quad \text{if} \quad -\bar{b} \equiv 2\Delta_1 - \Omega < 0. \end{aligned}$$

Since $-g_2 < 0$, there is no localized state for maxons. The spectral intensity near $\Omega = 2\Delta_1$ is then given by

$$I_{\epsilon_0 \epsilon_0}^{(2)}(\Omega)_{\max} = J(p_1) \frac{B_1 \sqrt{\bar{e}}}{e^{-2} + g_2 B_1^2} \theta(2\Delta_1 - \Omega) \quad (4.21)$$

The attractive interaction enhances the roton peak but depresses the maxon peak. This qualitative difference between rotons and maxons arises from the fact that the roton effective mass μ_0 is positive and the maxon effective mass μ_1 is negative.

If the roton has a finite lifetime, the δ -function in Eq. (4.19) is replaced by

$$\frac{\gamma_2}{\pi} \frac{1}{(\Omega - 2\Delta_0 + b)^2 + \gamma_2^2} \quad (4.22)$$

where γ_2 is the line width of the roton peak. The measured value of γ_2 (8) is $2.10^{-10} n_r \text{ sec}^{-1} \text{ cm}^3$ and n_r is the roton number density. We can incorporate the effect of finite roton lifetime into our calculation by redefining the one-particle Green's function

$$G_1(\vec{q}, \omega) = (\omega - \epsilon_q + i\gamma_r)^{-1} \quad (4.23)$$

where γ_1 is the single roton line width and ϵ_q is the roton energy. It can be shown that if $2\gamma_r \approx b$, $\gamma_2 \approx 2\gamma_1$. This gives the single roton line width $\gamma_r \approx 1.10^{-10} n_r \text{ sec}^{-1} \text{ cm}^3$. We will estimate γ_r in Section 6.

5. Roton-Roton Interactions

A physical picture of a roton has been proposed by Feynman⁽⁴⁾ and Feynman and Cohen⁽⁵⁾. Roughly it can be described as a small vortex ring of size comparable with the interparticle spacing. There is just space for an atom to pass through the center of the ring. Far from the ring there is a dipolar backflow field which is usually taken in hydrodynamic form. The backflow corresponds to a slow drift of atoms outside the ring which are returning for another passage through it.

A somewhat different picture of a roton has been proposed by Miller, Pines and Nozières⁽¹⁸⁾. They first considered the motion of an impurity atom in a Bose gas and calculated its coupling to the phonons in the Bose gas. They found that the wave function of the impurity atom is of the Feynman-Cohen type. The backflow acts to cancel the impurity current and increases the effective mass of the impurity atom. If we now assume that the Feynman excitation of Eq. (1.1) couples to the phonons in the same way as does an impurity atom, the result of including backflow would be to change the energy to

$$\epsilon_k^* = \frac{k^2}{2m^* S(k)}$$

where m^* is the effective mass of a helium atom. When the number of phonon modes is maximum and equal to the number of

atoms, m^* is equal to $3/2 m$ (m is the mass of a helium atom). Then the energy of the excitation is given by $\epsilon_k^* = k^2/3m S(k)$. ϵ_k^* provides a good approximation to the Feynman-Cohen calculation throughout the momentum region 1\AA^{-1} to 2\AA^{-1} . Now the Feynman excitation of Eq. (1.1) is very nearly a free particle excitation. It differs slightly in that it possesses a $1/3$ higher effective mass, which arises from the interaction with its nearest neighbors. Then we may visualize a roton as a modified free particle excitation which moves in liquid helium surrounded by a cloud of phonons, the latter describing the backflow of the other atoms at long distances from the atom in question.

In both pictures the flow pattern far from the center of the roton is of the dipolar form. But near the center we do not have an accurate description in either picture. The detailed structure of the roton core is probably not important in determining the energy of a roton and the Feynman-Cohen wave function of Eq. (1.2) gives a reasonable value for the energy of a roton.

We therefore will discuss the long range and short range interaction separately. We expect the long range roton-roton interaction V_L which arises from the interaction between their backflows to be a dipolar interaction. Then V_L will be attractive for certain orientations of the roton momenta \vec{p}_1 and \vec{p}_2 . As we understand less about the roton core structure, we will only indicate that the short separation roton-roton

interaction V_s is repulsive. We expect V_s to be strongly repulsive and with a range b comparable to the diameter of a helium atom.

5.1 Long Range Interaction V_L

(A) Feynman-Cohen Model

Suppose we picture a roton as a quasiparticle consisting of a core and its backflow. At a large distance from the core the influence of the roton on other excitations is via the backflow. The velocity field v_b of the backflow is given by

$$\vec{v}_b = -\frac{A_0}{m} \frac{\vec{p}_1}{r^3} (\hat{I} - 3\hat{r}\hat{r}) \quad (5.1)$$

where $A_0 = -3.6 \text{ \AA}^3$, m = mass of a helium atom and \vec{p}_1 is the roton momentum. The energy change ΔE of an excitation with momentum \vec{p}_2 in this velocity field is simply $\vec{p}_2 \cdot \vec{v}_b$:

$$\Delta E = v_{p_1 p_2}(\vec{r}) = -\frac{A_0}{m} \left[\frac{\vec{p}_1 \cdot \vec{p}_2}{r^3} - \frac{3(\vec{p}_1 \cdot \vec{r})(\vec{p}_2 \cdot \vec{r})}{r^5} \right] \quad (5.2)$$

We take $v_{p_1 p_2}(\vec{r})$ as the long range interaction between two rotors with momenta p_1 and p_2 . At interatomic distances, the hydrodynamic approximation breaks down and Eq. (5.2) will not be valid for $r < a$, the radius of the core.

(B) Hydrodynamic Model ⁽¹⁹⁾

Suppose we ignore the core structure and treat a roton as a hard sphere with mass m and radius a . Then we should be able to calculate the long range roton interaction by considering the backflow interaction between two hard spheres moving in an ideal fluid. The velocity potential $\phi(\vec{r})$ of one sphere moving with velocity \vec{u}_1 in the fluid is

$$\phi(\vec{r}) = \vec{\mu}_1 \cdot \vec{r}/r^3 \quad (5.3)$$

where $\vec{\mu}_1$ is determined by the boundary condition

$$-\left(\frac{\partial\phi}{\partial r}\right)_a = \vec{\mu}_1 \cdot \vec{r}/r \quad .$$

where a is the radius of the sphere. The dipole moment $\vec{\mu}_1$ is then equal to $\frac{a^3}{2} \vec{u}_1$.

The velocity potential ϕ of two spheres moving with velocities \vec{u}_1 and \vec{u}_2 in an ideal fluid is more complicated. We can determine ϕ approximation by requiring that the boundary conditions be satisfied to the order a^3/R^3 where R is the distance between the two spheres. Let us assume ϕ has the form

$$\phi = \frac{\vec{\mu}_1 \cdot \vec{r}}{r^3} + \frac{\vec{\mu}_2 \cdot \vec{r}'}{r^3} + \phi_1 + \phi_2 \quad (5.4)$$

where \vec{r} and \vec{r}' are spherical coordinates with origin at the centers of spheres 1 and 2, respectively. This is shown in Fig. 2 . The boundary conditions are

$$-\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \vec{\mu}_1 \cdot \vec{r}/r$$

and

$$-\left(\frac{\partial\phi}{\partial r'}\right)_{r'=a} = \vec{\mu}_2 \cdot \vec{r}'/r'.$$

Let us assume

$$\phi_1 = A_1 \frac{\vec{\mu}_1 \cdot \vec{r}'}{r'^3} + A_2 \frac{\vec{R} \cdot \vec{r}'}{r'^3} \quad (5.5)$$

$$\phi_2 = B_1 \frac{\vec{\mu}_2 \cdot \vec{r}}{r^3} + B_2 \frac{\vec{R} \cdot \vec{r}}{r^3} . \quad (5.6)$$

The boundary conditions will be satisfied to the order a^3/R^3 if

$$A_1 = B_1 = \frac{a^3}{2} \frac{1}{R^3} \quad (5.7)$$

$$A_2 = -\frac{3}{2} \frac{a^3}{R^5} (\vec{\mu}_1 \cdot \vec{R}) \quad (5.8)$$

$$B_2 = -\frac{3}{2} \frac{a^3}{R^5} (\vec{\mu}_2 \cdot \vec{R}) . \quad (5.9)$$

Then the complete potential is approximately

$$\phi = \bar{\phi}_1 + \bar{\phi}_2$$

with

$$\bar{\phi}_1 = \frac{\vec{\mu}_1 \cdot \vec{r}}{r^3} + \frac{a^3}{2} \frac{1}{R^3} \left[\frac{\vec{\mu}_1 \cdot \vec{r}'}{r'^3} - \frac{3(\vec{\mu}_1 \cdot \vec{R})(\vec{R} \cdot \vec{r}')}{R^2 r'^3} \right] \quad (5.10)$$

$$\bar{\phi}_2 = \frac{\vec{\mu}_2 \cdot \vec{r}'}{r'^3} + \frac{a^3}{2} \frac{1}{R^3} \left[\frac{\vec{\mu}_2 \cdot \vec{r}}{r^3} - \frac{3(\vec{\mu}_2 \cdot \vec{R})(\vec{R} \cdot \vec{r})}{R^2 r^3} \right] \quad (5.11)$$

The energy of the fluid is

$$\begin{aligned} E &= \frac{1}{2} \rho_o \int (\nabla \phi)^2 d^3r \\ &= \frac{1}{2} \rho_o \int_{\sigma_1 + \sigma_2} \phi \nabla \phi \cdot d\vec{s} \end{aligned}$$

where σ_1 and σ_2 are the surfaces of spheres 1 and 2, respectively.

We obtain for the energy of the fluid

$$\begin{aligned} E &= \frac{4\pi}{3} \rho_o \left(\frac{\mu_1^2}{a^3} + \frac{\mu_2^2}{a^3} \right) + 4\pi \rho_o \left[\frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{R^3} - \frac{3(\vec{\mu}_1 \cdot \vec{R})(\vec{\mu}_2 \cdot \vec{R})}{R^5} \right] \\ &= \frac{1}{2} m (\mu_1^2 + \mu_2^2) + \frac{\rho_o a^6}{R^3} \vec{\mu}_1 \cdot \vec{\alpha} \cdot \vec{\mu}_2 \end{aligned} \quad (5.12)$$

where $m = \frac{1}{2} \left(\frac{4\pi}{3} \rho_o a^3 \right)$ and $\vec{\alpha} = (\vec{1} - 3\hat{R}\hat{R})/R^3$.

The momenta \vec{p}_1 and \vec{p}_2 are given by

$$\vec{p}_1 = \frac{\partial E}{\partial \vec{\mu}_1} = m \vec{\mu}_1 + \pi \rho_0 a^6 \vec{\alpha} \cdot \vec{\mu}_2 .$$

$$\vec{p}_2 = \frac{\partial E}{\partial \vec{\mu}_2} = m \vec{\mu}_2 + \pi \rho_0 a^6 \vec{\alpha} \cdot \vec{\mu}_1 .$$

Then the Hamiltonian of the two spheres is

$$\begin{aligned} H &= \vec{p}_1 \cdot \vec{\mu}_1 + \vec{p}_2 \cdot \vec{\mu}_2 - E \\ &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{3a^3}{2m} \vec{p}_1 \cdot \vec{\alpha} \cdot \vec{p}_2 . \end{aligned} \quad (5.13)$$

We can identify the last term as the interaction between the spheres

$$V_{p_1 p_2}(\vec{r}) = -\frac{A}{m} \vec{p}_1 \cdot \vec{\alpha} \cdot \vec{p}_2 \quad (5.14)$$

with $A = 3/2 a^3$. If we put in the value of $A = -3.6 \text{ \AA}^3$ and assume m to be the mass of the helium atom, then $a = 1.33 \text{ \AA}$, which may be considered as an estimate of the radius of the roton core. We note that a is comparable with the radius of a helium atom.

(C) Phonon Exchange Between Two Rotons

We have shown from hydrodynamic considerations that the long range interaction $V_L(\vec{r})$ is a dipolar interaction. We can arrive at the same interaction if we consider the interaction of two rotons by phonon exchange. The interaction Hamiltonian H' between a roton with momentum \vec{p} and the velocity field \vec{v} of a phonon can be written as

$$H' = \frac{1}{2} \int \psi^\dagger(\vec{r}) [\vec{p} \cdot \vec{v} + \vec{v} \cdot \vec{p}] \psi(\vec{r}) d^3r \quad (5.15)$$

where ψ and ψ^\dagger are roton field operators. From equations (4.4a) and (A.12) we obtain

$$H' = \frac{i}{2} \sum_{\substack{\vec{p} \\ \vec{q}}} \frac{\omega_q}{2n_0 m q^2} (2\vec{p} \cdot \vec{q} + q^2) b_{\vec{p}+\vec{q}} b_{\vec{p}} (a_{-\vec{q}}^\dagger + a_{\vec{q}}) \quad (5.16)$$

where $a_{\vec{q}}^\dagger$, $a_{\vec{q}}$, $b_{\vec{p}}^\dagger$ and $b_{\vec{p}}$ are the phonon creation and annihilation operators, and the roton creation and annihilation operators, respectively. The q^2 term in Eq. (5.16) may be neglected if we consider only long wavelength phonons. We now make a canonical transformation in order to eliminate the phonon coordinates to lowest order. The transformed Hamiltonian is

$$\tilde{H} = e^{-S} (H_0 + \lambda H') e^S \quad (5.17)$$

H_0 is the total Hamiltonian of the noninteracting rotons and phonons, and λ is the expansion parameters. The transformed Hamiltonian H' is then approximately

$$\tilde{H} = H_0 + \frac{\lambda}{2} [H', S] + O(\lambda^3) = H_0 + \tilde{H}' \quad (5.19)$$

The matrix element of S between two unperturbed states $|m\rangle$ and $|n\rangle$ is given by

$$\langle n|S|m\rangle = \langle n|H'|m\rangle / (E_m - E_n)$$

where

$$H_0|m\rangle = E_m|m\rangle$$

We eliminate the phonon coordinates by taking the matrix element of \tilde{H}' in the phonon ground state $|0\rangle_{ph}$ and obtain

$$H'_{eff} = \langle 0 | \tilde{H}' | 0 \rangle_{ph} = - \sum_{\vec{p}_1 \vec{p}_2 \vec{q}} \frac{(\vec{p}_1 \cdot \vec{q})(\vec{p}_2 \cdot \vec{q})}{q^2} b_{\vec{p}_1 + \vec{q}}^+ b_{\vec{p}_1} b_{\vec{p}_2 - \vec{q}}^+ b_{\vec{p}_2} \quad (5.20)$$

The effective potential between two rotons is then

$$V_{p_1 p_2}(\vec{q}) = - \frac{1}{n_0 m} \frac{(\vec{p}_1 \cdot \vec{q})(\vec{p}_2 \cdot \vec{q})}{q^2} \quad (5.21)$$

The Fourier transform of $V_{p_1 p_2}(\vec{q})$ is, when $\vec{r} > 0$,

$$V_{p_1 p_2}(\vec{r}) = -\frac{1}{4\pi n_0 m} \left(\frac{\vec{p}_1 \cdot \vec{p}_2}{r^3} - 3 \frac{(\vec{p}_1 \cdot \vec{r})(\vec{p}_2 \cdot \vec{r})}{r^5} \right), (r > 0). \quad (5.22)$$

This is exactly the same as given by Eq. (5.2) with

$$A = -1/4\pi n_0 = -3.6 \text{ \AA}^3.$$

We conclude that the long range roton-roton interaction is a dipolar interaction. All three methods lead to same form of interaction. This form is valid only $r > a$, the radius of the roton core. For a pair of rotons with zero total momentum, the interaction is attractive if the relative momentum is parallel to the relative position vector r and is repulsive if they are perpendicular. Now the question is whether two rotons can form a bound state of d-symmetry when they interact via the dipolar potential. We calculate the binding energy of the d-state by a variational method. The trial wave function of d-symmetry is taken to be, for zero total momentum,

$$\psi_d(r) = \sum_p e^{i \vec{p} \cdot (\vec{r}_1 - \vec{r}_2)} h_p Y_2^0(\hat{p}) \quad (5.23)$$

where $h_p = 1/(p-p_0)^2 + \alpha^2$ with α as the variational parameter, \vec{r}_1 and \vec{r}_2 are the coordinates of the two rotons, and Y_2^0 is the spherical harmonic of order 2. This choice of trial wave function gives us the simplest localized two roton wave packet. The potential $V_L(\vec{r})$ is set equal to zero for $r < r_0$, where the back-flow picture is no longer valid. The Hamiltonian of the interacting roton pair with zero total momentum is

$$H = 2\Delta_0 + T(\vec{p}) + V_L(\vec{r}) + V_S(\vec{r}) \quad (5.24)$$

where

$$T = (p - p_0)^2/\mu_0, \quad V_L = \frac{p_0^2}{2\pi m_0 m} \frac{P_2(\hat{p} \cdot \hat{r})}{r^3}$$

with $P_2(\hat{p} \cdot \hat{r}) = \frac{1}{2} (3 \cos \theta_{\vec{p} \cdot \vec{r}} - 1)$ and V_S is the short range interaction. We expect the short range part of the potential to be repulsive. However, it is not important for the calculation of the binding energy of 2 rotons in a d-state as the wave function is small as $r \rightarrow 0$. The normalization of the wave function ψ_d is easily evaluated

$$I = \int |\psi_d|^2 d^3r = \left(\frac{p_0}{4\pi}\right)^2 \frac{1}{\alpha^3} \quad (5.25)$$

The average values of the kinetic energy $\langle T \rangle$ and the potential energy $\langle V \rangle$ are

$$\langle T \rangle = \frac{1}{I} \int \psi_d^* \frac{(p-p_0)^2}{\mu_0} \psi_d d^3r = \frac{\alpha^2}{\mu_0} ; \quad (5.26)$$

$$\begin{aligned} \langle V \rangle &= \langle V_L \rangle = \frac{1}{I} \int_{r_0}^{\infty} \psi_d^* V_L(r) \psi_d d^3r \\ &\approx - \frac{p_0^2}{2\pi n_0 m} \frac{e^{-2\alpha r_0}}{r_0^2} \alpha . \end{aligned} \quad (5.27)$$

We can now minimize the energy

$$E = 2\Delta_0 + \langle T \rangle + \langle V \rangle$$

by varying α and obtain the minimum value of E_m of E

$$E_m = 2\Delta_0 - \frac{\alpha^2}{\mu} \left(\frac{1 + 2\alpha r_0}{1 - 2\alpha r_0} \right) , \quad (5.28)$$

where α satisfies

$$\frac{2\alpha}{\mu} = \left(\frac{p_0}{r_0} \right)^2 \frac{e^{-2\alpha r_0}}{2\pi n_0 m} (1 - 2\alpha r_0) . \quad (5.29)$$

Using the numerical values

$$p_0 = 1.9 \text{ \AA}^{-1}$$

$$r_0 = 4.0 \text{ \AA}$$

$$\mu_0 = 1.10^{-24} \text{ gm}$$

we find

$$\alpha = 0.05 \text{ \AA}^{-1}$$

and

$$E_m = 2\Delta_0 = -0.40^\circ\text{K}$$

The value of α^{-1} gives an estimate of the size R of the bound roton pair. $R \sim \alpha^{-1} \simeq 20 \text{ \AA}$. This compares well with the experimental value of the boundary energy found by Greytak et al. (9)

We can also calculate the binding energy when the total momentum \vec{P} is not zero. For finite \vec{P} we use the trial wave function

$$\psi_d = e^{i\vec{P}\cdot\vec{R}} \sum_{\vec{p}} e^{i\vec{p}\cdot\vec{r}} Y_2^0(\vec{p}) h_{\vec{p}} \quad (5.30)$$

and the function

$$h_{\vec{p}} = \frac{1}{(p - p_0 - \delta)^2 + \alpha^2}$$

with two variation parameters α and δ . We consider only the case $|\vec{P}| \ll p \approx p_0$. After a similar calculation we obtain

$$E_m = 2\Delta_0 + \frac{p^2}{8\mu_c} - \frac{\alpha^2}{\mu_0} \left(\frac{1 + 2\alpha r_0}{1 - 2\alpha r_0} \right). \quad (5.31)$$

This means for small P the kinetic energy increases as $p^2/8\mu_c$. At $P \approx 5\alpha$ the binding energy vanishes. In Section 6.1 we will show that the binding energy $b_{\vec{p}}$ is a rapidly decreasing function of P . At $P \approx 5\alpha$ $b_{\vec{p}} \approx 0.02^\circ\text{K}$. Therefore no bound state exists unless the roton momenta \vec{p}_1 and \vec{p}_2 are almost antiparallel.

5.2 Short Range Interaction $V_S(\vec{r})$

The interaction between two rotons at a short distance depends on the roton structure near its core. As the core structure is less understood, the explicit form of the short range interaction $V_S(\vec{r})$ cannot be determined. We can indicate plausibly that the interaction between two rotons is repulsive at short distances from the Feynman wave function ψ^S . We use the Feynman wave function ψ^S instead of the Feynman-Cohen wave function ψ because the backflow is negligible near the roton core.

The Feynman wave function for a roton with momentum \vec{p} is given by

$$\psi_p^S = F_p \phi_0 = \sum_i e^{i\vec{p} \cdot \vec{r}_i} \phi_0$$

where ϕ_0 is the ground state wave function and the sum is over all helium atoms. When $\vec{p} = 0$, ψ_p^s reduces to the unnormalized ground state wave function $\sum_i \phi_0$. The Feynman wave function for two rotons with momenta \vec{p}_1 and \vec{p}_2 must be orthogonal to all ψ_p^s . We write the Feynman wave function for two rotons as

$$\psi_{\vec{p}_1, \vec{p}_2}^s = \sum_i \sum_j e^{i\vec{p}_1 \cdot \vec{r}_i + i\vec{p}_2 \cdot \vec{r}_j} \phi_0 - \sum_{\vec{p}} a_{\vec{p}} \psi_{\vec{p}}^s \quad (5.32)$$

where $a_{\vec{p}}$'s are constants to be determined by requiring that $\psi_{\vec{p}_1, \vec{p}_2}^s$ is orthogonal to all $\psi_{\vec{p}}^s$. The condition for two wave functions to be orthogonal is $\langle \psi_{\vec{p}}^s | \psi_{\vec{p}_1, \vec{p}_2}^s \rangle = 0$. Now

$$\begin{aligned} \langle \psi_{\vec{p}}^s | \psi_{\vec{p}_1, \vec{p}_2}^s \rangle &= \sum_i \sum_j \sum_k \int e^{i\vec{p}_1 \cdot \vec{r}_i + i\vec{p}_2 \cdot \vec{r}_j - i\vec{p} \cdot \vec{r}_k} \phi_0^2 d^3r_1 \dots d^3r_n \\ &\quad - \sum_{\vec{p}_1} a_{\vec{p}_1} \sum_i \sum_k \int e^{i\vec{p}_1 \cdot \vec{r}_i - i\vec{p} \cdot \vec{r}_k} \phi_0^2 d^3r_1 \dots d^3r_n \\ &= \int \rho_3(\vec{r}_1, \vec{r}_2, \vec{r}_3) e^{i\vec{p}_1 \cdot \vec{r}_1 + i\vec{p}_2 \cdot \vec{r}_2 - i\vec{p} \cdot \vec{r}_3} d^3r_1 d^3r_2 d^3r_3 \\ &\quad - \sum_{\vec{p}_1} a_{\vec{p}_1} \int \rho_2(\vec{r}_1, \vec{r}_3) e^{i\vec{p}_1 \cdot \vec{r}_1 - i\vec{p} \cdot \vec{r}_3} d^3r_1 d^3r_3, \end{aligned} \quad (5.33)$$

where $\rho_2(\vec{r}_1, \vec{r}_2)$ is the probability of finding an atom at \vec{r}_1 and another atom at \vec{r}_2 , and $\rho_3(r_1, r_2, r_3)$ is the probability of finding one atom at each of the three points \vec{r}_1, \vec{r}_2 and \vec{r}_3 . The function $\rho_2(\vec{r}_1, \vec{r}_2) = \rho_0 p(\vec{r}_1 - \vec{r}_2)$ where ρ_0 is the density. $p(\vec{r}_1 - \vec{r}_2)$ is the pair correlation function and its Fourier transform is the liquid structure factor $S(k)$,

$$S(k) = \int p(\vec{r}) e^{i\vec{k} \cdot \vec{r}} d^3r \quad (5.34)$$

Therefore the second term of Eq. (5.33)

$$- \sum_{\vec{p}_1} a_{\vec{p}_1} \int_{\vec{p}_1} \rho_0 p(\vec{r}_{13}) e^{i(\vec{p}_1 - \vec{p}) \cdot \vec{r}_1 + i\vec{p} \cdot (\vec{r}_1 - \vec{r}_3)} d^3r_1 d^3r_{13} \quad (5.35)$$

$$= - a_{\vec{p}} S(\vec{p})$$

The largest contribution to the function $\rho_3(r_1, r_2, r_3)$ comes from $\rho_2(r_1, r_2) p(r_1, r_3)$. If we make the superposition approximation

$$\rho_3(r_1, r_2, r_3) \approx \rho_2(r_1, r_2) p(r_1, r_3) \quad (5.36)$$

the first term in Eq. (5.33)

$$\begin{aligned}
 & \int \rho_3(\vec{r}_1, \vec{r}_2, \vec{r}_3) e^{i\vec{p}_1 \cdot \vec{r}_1 + i\vec{p}_2 \cdot \vec{r}_2 - i\vec{p} \cdot \vec{r}_3} d^3r_1 d^3r_2 d^3r_3 \\
 &= \int \rho_2(\vec{r}_1 - \vec{r}_2) \rho(\vec{r}_1 - \vec{r}_3) e^{i(\vec{p}_1 + \vec{p}_2 - \vec{p}) \cdot \vec{r}_1 + i\vec{p}_2 \cdot \vec{r}_{21} + i\vec{p} \cdot \vec{r}_{13}} \\
 & \quad d^3r_1 d^3r_{12} d^3r_{13} \\
 &= \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}} S(\vec{p}_2) S(\vec{p}) \quad . \quad (5.37)
 \end{aligned}$$

Thus Eq. (5.33) becomes

$$0 = \left(\delta_{\vec{p}_1 + \vec{p}_2, \vec{p}} S(\vec{p}_2) - a_p \right) S(\vec{p}) \quad . \quad (5.38)$$

Since $S(p)$ is finite,

$$a_p = S(p_2) \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}} \quad .$$

For roton momentum P_2 , $S(p_2) \approx 1.3$. Therefore, we obtain the Feynman wave function for two rotons

$$\psi_{P_1, P_2}^S = \sum_i \sum_{j \neq i} e^{i\vec{p}_1 \cdot \vec{r}_i + i\vec{p}_2 \cdot \vec{r}_j} \phi_0 - (S(p_2) - 1) \psi_{P_1 + P_2}^S \quad .$$

This wave function tells us that the roton momenta \vec{p}_1 and \vec{p}_2 are mainly carried by different atoms \vec{r}_i and \vec{r}_j . As the i^{th} atom approaches the j^{th} atom, $\psi_{\vec{p}_1, \vec{p}_2}^s$ vanishes as a result of ϕ_0 being zero if any two atoms overlap. The repulsive and short range interaction between two helium atoms gives rise to the short range repulsive interaction V_S between two rotors. As we do not have an accurate description of the roton core structure, the explicit form of V_S cannot be determined. We will assume that it has a strength V_0 and a range b . Our consideration indicates that b is approximately the diameter of a helium atom.

In the next section we will investigate roton-roton scattering using several different short range repulsive potentials. It will be shown that the predicted roton-roton collision frequency agrees with the experimental value if V_0 is strong enough and b is comparable with the diameter of a helium atom.

6. Roton-Roton Scattering

Information about roton-roton interactions may be obtained from viscosity η of He II and from roton line widths as measured by Raman scattering. The temperature independent part of η is interpreted by Landau and Khalatnikov⁽¹⁵⁾ to be the viscosity η_r of the roton gas. If τ_1 is the relaxation time of the roton gas, the viscosity η_r can be shown to be

$$\eta_r = \frac{p_0^2 \tau_1}{15 \mu_0} n_r$$

where n_r is the roton number density. Brewer and Edwards⁽¹⁴⁾ have measured η_r and determined

$$\tau_1^{-1} = 2.10^{-10} n_r. \quad (6.1)$$

The roton line width δE can be interpreted as a result of the scattering of the rotors created by other thermally excited rotors⁽⁸⁾. The line width leads to a relaxation time τ_2 which has the value

$$\tau_2^{-1} = 1.10^{-10} n_r.$$

Thus τ_1^{-1} and τ_2^{-1} are approximately equal. We want to calculate τ^{-1} and compare it with the experimental τ_1^{-1} . We adopt a simple point of view that the collision frequency τ^{-1} of a roton with all other rotons is the sum of the collision frequency of a roton with another roton,

$$\tau^{-1} = \sum_{\vec{p}_2} \omega(\vec{p}_1, \vec{p}_2) n_{\vec{p}_2}$$

where $\omega(\vec{p}_1, \vec{p}_2)$ is the transition probability per unit time from two roton state $|\vec{p}_1, \vec{p}_2\rangle$ to all other two roton states.

$$\omega(\vec{p}_1, \vec{p}_2) = \pi \sum_{\vec{p}'_1, \vec{p}'_2} |\langle \vec{p}'_1, \vec{p}'_2 | T | \vec{p}_1, \vec{p}_2 \rangle|^2 \delta(E_1 + E_2 - E'_1 - E'_2) \quad (6.2)$$

$n_{\vec{p}_2}$ is the roton distribution function. Then the collision frequency τ^{-1} is approximately equal to

$$\tau^{-1} = \pi \sum_{\vec{p}'_1, \vec{p}'_2, \vec{p}_2} |\langle \vec{p}'_1, \vec{p}'_2 | T | \vec{p}_1, \vec{p}_2 \rangle|^2 \delta(E_1 + E_2 - E'_1 - E'_2) n_{\vec{p}_2} \quad (6.3)$$

In general τ^{-1} of Eq. (6.3) is not exactly equal to the collision frequency used in calculating viscosity. In fact we need a different τ^{-1} for each transport process. For the viscosity calculation we should subtract from Eq. (6.3) a part due to forward and backward scattering. This means that the

correct τ^{-1} should be smaller. Thus our expression for τ^{-1} as given by Eq. (6.3) may be viewed as an upper limit of τ_1^{-1} used in calculating viscosity η_r . A similar consideration also gives τ^{-1} as an upper limit of τ_2^{-1} of roton line width. As the exact form of the repulsive interaction between rotons is not known, we will assume several different potentials for the roton-roton interaction to calculate τ^{-1} .

6.1 Pseudopotential $V(r) = V_0(r)$

In 1949 Landau and Khalatnikov⁽¹⁵⁾ were interested in the temperature dependence of τ^{-1} . They assume a pseudopotential $V(\vec{r}) = V_0 \delta(\vec{r})$ for the roton-roton interaction and evaluated the matrix element $\langle \vec{p}'_1 \vec{p}'_2 | T | \vec{p}_1 \vec{p}_2 \rangle$ by Born approximation. Their results are

$$\langle \vec{p}'_1 \vec{p}'_2 | T | \vec{p}_1 \vec{p}_2 \rangle = 2V_0 \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}'_1 + \vec{p}'_2} \quad (6.4)$$

and

$$\tau^{-1} = 4\pi |V_0|^2 \sum_{\vec{p}_2} n_{\vec{p}_2} \sum_{\vec{p}'_1} \{ \delta(\epsilon(\vec{p}_1) + \epsilon(\vec{p}_2) - \epsilon(\vec{p}'_1) - \epsilon(\vec{P} - \vec{p}'_1)) \} \quad (6.5)$$

where $\vec{P} = \vec{p}_1 + \vec{p}_2$ and $\epsilon(\vec{p})$ is the roton energy. The density of states

$$F(\vec{p}) = \sum_{\vec{p}'} \delta(\epsilon(\vec{p}_1) + \epsilon(\vec{p}_2) - \epsilon(\vec{p}'_1) - \epsilon(\vec{P} - \vec{p}'_1)) \quad (6.6)$$

is evaluated in Appendix B and is equal to $\mu_0 p_0^2 / 2\pi P$, where p_0 and μ_0 are roton parameters. This result can also be obtained by the following simple consideration. Let $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$ and $\vec{p}' = (\vec{p}'_1 - \vec{p}'_2)/2$ be the relative momenta before and after the collision. By the conservation of energy, \vec{p}' must be very near to the circle generated by rotating \vec{p} about the total momentum \vec{P} . Thus the allowed states lie within a torus with a small cross section. The volume of this torus is equal to $F(\vec{P})$.

The collision frequency τ_{LK}^{-1} , after taking the average of \vec{P} , is given by

$$\tau_{LK}^{-1} = 2\mu_0 p_0 |V_0|^2 n_r \quad (6.7)$$

This differs from the result obtained by Landau and Khalatnikov by a factor of 2. V_0 can be determined by a comparison between equations (6.1) and (6.7) and is equal to 2.10^{-38} erg cm³. This value of V_0 is an order of magnitude larger than g_2 of the separable potential of Eq. (4.14)

The above calculation is based on Born approximation. We can show that the Born approximation is not valid in this case. Indeed we can show that for a pseudopotential $V_0 \delta(\vec{r})$ there is an upper bound for τ^{-1} which is independent of V_0 and is 4 times smaller than the experimental value of τ^{-1} .

By the optical theorem we can rewrite Eq. (6.3) as

$$\tau^{-1} = - \sum_{\vec{p}_2} \text{Im} \langle \vec{p}_1 \vec{p}_2 | T | \vec{p}_1 \vec{p}_2 \rangle n_{\vec{p}_2} \quad (6.8)$$

The matrix element $\langle \vec{p}_1 \vec{p}_2 | T | \vec{p}_1 \vec{p}_2 \rangle$ for a pseudopotential $V_0 \delta(r)$ is given by

$$\langle \vec{p}_1 \vec{p}_2 | T | \vec{p}_1 \vec{p}_2 \rangle = \frac{2V_0}{1 - V_0 F(\vec{P}, \Omega)} \quad (6.9)$$

$$\text{where } F(\vec{P}, \Omega) = \sum_{\vec{p}} (\Omega - \epsilon(\vec{p}) - \epsilon(\vec{P} - \vec{p}) + i\gamma)^{-1} \quad (6.10)$$

If we write F in terms of its real and imaginary part,

$F = F_1 + i F_2$, then

$$\text{Im} \langle \vec{p}_1 \vec{p}_2 | T | \vec{p}_1 \vec{p}_2 \rangle = \frac{2V_0^2 F_2}{(1 - V_0 F_1)^2 + V_0^2 F_2^2} \quad (6.11)$$

An upper bound of $\text{Im} \langle p_1 p_2 | T | p_1 p_2 \rangle$ is $2/F_2$, which is independent of V_0 . F_2 is the density of states, and is equal to $\mu_0 p_0^2 / 2P$.

Then the upper limit $\tilde{\tau}^{-1}$ has the value

$$\begin{aligned} \tilde{\tau}_{LK}^{-1} &= \frac{4}{\mu_0 p_0^2} \sum_{p_2} P n_{p_2} \\ &= \frac{32}{3\mu_0 p_0} n_r \\ &= 5.10^{-11} n_r \end{aligned} \quad (6.12)$$

which is 4 times smaller than the experimental value and is independent of V_0 .

To show that Born approximation is not valid, we compare the first and second term in the expansion of t-matrix of Eq. (6.11). The condition for Born approximation to be valid is

$$V_0 F_2 \ll 1 \quad (6.13)$$

or

$$\frac{V_0 \mu_0 p^2}{2P} \ll 1 \quad (6.14)$$

The smallest value of L.H.S. of Eq. (6.14) is 2. Therefore Born approximation is not valid.

If V_0 is negative, roton bound states exist. The binding energy $b_{\vec{p}}$ can be obtained from the pole of the t-matrix of Eq. (6.9). From Eqs. (B.3) and (B.5) we have

$$b_{\vec{p}} (\vec{P} = 0) = \frac{\mu_0 p^4 V_0^2}{4\pi^2} \quad (6.15)$$

$$b_{\vec{p}} (\vec{P} \neq 0) \doteq \frac{p^2}{\mu_0} \exp \left[\frac{-2\pi P}{\mu_0 p^2 V_0} \right] \quad (6.16)$$

Experimentally⁽⁹⁾ $b_{\vec{p}}(\vec{P} \neq 0) = 0.37^\circ\text{K}$. This corresponds to a $V_0 \approx 10^{-39}$ which is of the order of magnitude of g_2 . For finite P , $b_{\vec{p}}$ is a rapidly decreasing function of P . The average kinetic energy of rotons is kT . Therefore the bound states has very little effect on the scattering for $T > 1^\circ\text{K}$.

6.2 Dipole-Dipole Interaction

The contribution of the long range part of roton-roton interaction to the collision frequency has been calculated by Toigo⁽¹⁶⁾. The potential he used is

$$V = -\frac{A_0}{m} \left(\frac{\vec{p}_1 \cdot \vec{p}_2}{r^3} - \frac{3(\vec{p}_1 \cdot \vec{r})(\vec{p}_2 \cdot \vec{r})}{r^5} \right) p(r) \quad (6.17)$$

where $p(r)$ is the pair correlation function between atoms in the liquid. This potential automatically becomes zero when $r < r_0$, the diameter of helium atom. $r_0 = 2.4 \text{ \AA}$ is about the size of the roton core. At large r , $p(r)$ approaches 1 and V becomes the long range interaction V_L of Eq. (5.2). The t-matrix element is obtained by Born approximation

$$\begin{aligned} \langle \vec{p}_1' \vec{p}_2' | V | \vec{p}_1 \vec{p}_2 \rangle = & -\frac{4\pi}{m} A_0 \left[(\vec{p}_1 \cdot \vec{p}_2 - \frac{3(\vec{p}_1 \cdot \vec{q}_1)(\vec{p}_2 \cdot \vec{q}_1)}{q^2}) I(\vec{q}_1) \right. \\ & \left. + (\vec{p}_1 \cdot \vec{p}_2 - \frac{3(\vec{p}_1 \cdot \vec{q}_2)(\vec{p}_2 \cdot \vec{q}_2)}{q^2}) I(\vec{q}_2) \right] \end{aligned} \quad (6.18)$$

where $\vec{q}_1 = \vec{p}_1 - \vec{p}'_1$ is a small vector while $\vec{q}_2 = \vec{p}_1 - \vec{p}'_2 \approx 2p_o \sin \theta/2$ is a large vector. θ is the angle between \vec{p}_1 and \vec{p}_2 .

$I(q_1)$ and $I(q_2)$ have the form

$$I(q) = \int_{r_o}^{\infty} j_2(qr) \frac{p(r)}{r} dr \quad (6.19)$$

$$\approx j_1(qr_o)/qr_o .$$

The transition probability per unit time $\omega(\vec{p}_1, \vec{p}_2)$, after integrating over \vec{q}_1 and \vec{q}_2 , is given by

$$\omega(\vec{p}_1, \vec{p}_2) = \frac{A_o^2 \mu_o p_o^5}{\pi m^2} F_1(\theta) \quad (6.20)$$

where F_1 is a function of the angle between \vec{p}_1 and \vec{p}_2 only. The collision frequency τ_T^{-1} is then obtained by averaging F_1 over all angles. Toigo found that τ_T^{-1} is 20 times smaller than the experimental value. Thus the long range part of roton-roton interaction is negligible.

We note that Born approximation is valid in Toigo's calculation as the strength of the potential is an order of magnitude smaller than the strength of the pseudopotential used by Landau and Khalatnikov.

6.3 Variational Method

Roton-roton scattering differs from ordinary particle-particle scattering in one important aspect: in ordinary particle-particle scattering, the center of mass motion and the relative motion can be separated and the problem is reduced to a one particle problem, whereas in roton-roton scattering these two motions are coupled. This coupling of the center of mass motion and relative motion comes from the peculiarity of the roton energy spectrum. The phenomenological Hamiltonian of two interacting rotons is given by

$$\begin{aligned}
 H &= 2\Delta_o + (p_1 - p_o)^2/2\mu_o + (p_2 - p_o)^2/2\mu_o + V(r) \\
 &= 2\Delta_o + (|\vec{p} + \vec{P}/2| - p_o)^2/2\mu_o + (|\vec{p} - \vec{P}/2| - p_o)^2/2\mu_o + V(r)
 \end{aligned}
 \tag{6.21}$$

where $\vec{P} = \vec{p}_1 + \vec{p}_2$ and $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$. This coupling of center of mass and relative motion prevents us from applying the method of partial wave expansions to roton-roton scattering. Other approximations such as Born approximation and the Eikonal approximation are also not suitable. We therefore calculate the t-matrix by the Lippmann-Schwinger variational method^(20,21). A stationary expression for $\langle \phi_b | T | \phi_a \rangle$ between two plane wave states $|\phi_a\rangle$ and $|\phi_b\rangle$ is given by

$$\begin{aligned}
 \langle \phi_b | T | \phi_a \rangle &= \langle \psi_b^{(-)} | V | \phi_a \rangle + \langle \phi_b | V | \psi_a^{(+)} \rangle - \\
 &\quad \langle \psi_b^{(-)} | V - V G_o^{(+)} V | \psi_a^{(+)} \rangle .
 \end{aligned}
 \tag{6.22}$$

$|\psi_a^{(+)}\rangle$ is the outgoing stationary wave state whose asymptotic form is the plane wave state $|\phi_a\rangle$ and $|\psi_b^{(-)}\rangle$ is the incoming stationary wave state whose asymptotic form is the plane wave state $|\phi_b\rangle$. $G_0^{(+)}$ is the retarded Green's function

$$G_0^{(+)} = (E - H_0 + i\gamma)^{-1} \quad (6.23)$$

where H_0 is the unperturbed Hamiltonian of the system. We choose the amplitude of the outgoing wave $\psi_a^{(+)}$ and the incoming wave $\psi_b^{(-)}$ as our variational parameters.

$$\psi_a^{(+)} = A \phi_a$$

$$\psi_b^{(-)} = B \phi_b$$

Equation (6.22) is stationary with respect to $\psi_a^{(+)}$ and $\psi_b^{(-)}$ and it provides an extremum value of the t-matrix. If we vary $\langle \phi_b | T | \phi_a \rangle$ with respect to A and B separately, we obtain an extremum value

$$\langle \phi_b | T | \phi_a \rangle_0 = \frac{\langle \phi_b | V | \phi_a \rangle^2}{\langle \phi_b | V - V G_0^{(+)} V | \phi_a \rangle} \quad (6.24)$$

In this approximation the second Born approximation appears in the denominator. Now we put $|\phi_a\rangle = |\vec{p}_1, \vec{p}_2\rangle$ and $|\phi_b\rangle = |\phi_a\rangle$, and we obtain

$$\langle k \vec{p}_1, \vec{p}_2 | T | \vec{p}_1, \vec{p}_2 \rangle = \frac{V^2(\vec{q} = 0)}{V(0) - G} \quad (6.25)$$

where

$$V(\vec{q}) = \langle \vec{p}_1 + \vec{q}, \vec{p}_2 - \vec{q} | T | \vec{p}_1, \vec{p}_2 \rangle$$

and

$$G = \sum_{\vec{q}} \frac{V^2(\vec{q})}{\Omega - \epsilon_{\vec{p}_1 + \vec{q}} - \epsilon_{\vec{p}_2 - \vec{q}} + i\gamma}$$

If we substitute G by its real and imaginary part G_1 and G_2 into Eq. (6.25), then

$$\langle \vec{p}_1 \vec{p}_2 | T | \vec{p}_1 \vec{p}_2 \rangle = \frac{V^2(0)}{(V(0) - G_1) + i G_2}$$

Before we go on, let us consider the following potentials with a range b and their Fourier transforms:

(A) Gaussian Potential

$$V(\vec{r}) = v_0 e^{-r^2/b^2},$$

$$V(\vec{q}) = V(0) e^{-q^2 b^2/4} \quad \text{with}$$

$$V(0) = v_0 b^3 \pi^{3/2}.$$

(B) Potential

$$V(\vec{r}) = V_0 / (r^2 + b^2)^2 ,$$

$$V(\vec{q}) = V(0) e^{-|q|b} \quad \text{with}$$

$$V(0) = V_0 \pi^2 / b .$$

(C) Soft-Sphere Potential

$$V(\vec{r}) = \begin{matrix} V_0 & \text{if } r \leq b \\ 0 & \text{if } r > 0 \end{matrix} ,$$

$$V(\vec{q}) = 3V(0) j_1(qb) / qb \quad \text{with}$$

$$V(0) = \frac{4\pi}{3} V_0 b^3 .$$

We observe that $V(\vec{q})$ is small if $q \gg \frac{1}{b}$. Therefore in calculating G we can make the approximation $q \ll p_1$ and $q \ll p_2$. We are interested in the imaginary part G_2 of G which is given by

$$G_2 = \pi \int_q V^2(\vec{q}) \delta(\Omega - \epsilon_{\vec{q}+\vec{p}_1} - \epsilon_{\vec{p}_2-\vec{q}}) \quad (6.27)$$

From Appendix C we have

$$G_2 \approx \frac{\mu_o p_o^2}{4\pi p P} \int_{-\infty}^{\infty} dq_z V^2(\mu k T + qz^2) \quad (6.28)$$

where we have put $\mu_o(\Omega - 2\Delta_o) \approx \mu k T$. We may neglect $\mu k T$ in comparison to qz^2 and obtain

$$G_2 \approx \frac{\mu_o p_o^2}{\alpha b p P} V^2(0) \quad (6.29)$$

where α is a constant which depends on the potential used. For potential (A) $\alpha = \sqrt{8\pi}$, (b) $\alpha = 4\pi$ and (C) $\alpha = 10/3$.

The collision frequency τ^{-1} is

$$\begin{aligned} \tau^{-1} &= \frac{4\alpha b}{\mu_o p_o^2} \int_{\vec{p}_2} p^P n_{p_2} \\ &= \frac{\pi\alpha b}{\mu_o} n_r . \end{aligned} \quad (6.30)$$

Here we have neglected $V(0)$ in comparison to G_2 . This is valid if $V(0) > \frac{\alpha b}{\mu_o} \sim \alpha 10^{-38}$ erg cm³. If we compare Eqs. (6.12) and (6.30), we find

$$\tau^{-1} = \left(\frac{3\pi \alpha b p_o}{32} \right) \tilde{\tau}_{LK}^{-1} . \quad (6.31)$$

Thus τ^{-1} and $\tilde{\tau}_{LK}^{-1}$ differ by a factor $p_0 b$, which is a measure of the maximum relative angular momentum. From the experimental value of $\tau^{-1} = 2.10^{-38} n_r$, we find the range b for potential (A) $b = 1.2 \text{ \AA}$, (B) $b = 0.6 \text{ \AA}$ and (C) $b = 2.1 \text{ \AA}$. These values of b are approximately equal to the diameter of a helium atom.

6.4 Discussion

We have shown that the pseudopotential of Landau and Khalatnikov does not give a correct value of τ^{-1} . This is because the momentum transfer $\Delta \vec{p} = \vec{q}$ can be large as $V(\vec{q}) = V(0)$ for all \vec{q} . It only leads to S-wave scattering (for $P = 0$). The correct value of τ^{-1} can be obtained only if the range of the potential is taken into account. In the case of a potential with range b , several angular momentum channels are involved in the scattering. Not all $V(\vec{q})$ are equally important. Only the contributions of $V(\vec{q})$ with $q < \frac{1}{b}$ are important. This limits the momentum transfer $q \ll p_0$ and the roton-roton scattering is confined to small angle scattering. This is consistent with the nature of the roton — an excitation with high momentum.

7. Conclusion

Raman scattering in He II is found to be a second order process in the polarizability α of a helium atom. The incident light polarizes an atom. The polarized atom interacts via dipole-dipole interaction with a second atom and induces a dipole in it, which then radiated the scattered light. This process leads to almost completely depolarized scattered light and almost isotropic distribution of the scattered light. These results are in good agreement with experiment.

The experimental spectral intensity of the scattered light has a single peak near $\omega_0 - \omega = 2\Delta_0$. This can be understood if the interaction between elementary excitations is taken into account. The roton peak is shown to be due to a roton bound state of d-symmetry. The spectral distribution of the scattered light is calculated using a pseudopotential for the interaction of rotons and qualitative agreement with experiment obtained.

The pseudopotential model is unsatisfactory to describe the interaction of rotons and cannot be used to explain both the bound states and scattering of 2 rotons. A more realistic model is constructed for the interaction of 2 rotons using the Feynman-Cohen model of a roton. It is shown that at large separations of the 2 rotons, the interaction has a dipolar form, which results from the interaction of the backflow fields of the rotons. At small separation the interaction of 2 rotons is shown to be repulsive.

A variational calculation of the binding energy of 2 rotons in a d-state is made using the above dipolar interaction. A bound state is obtained in reasonable agreement with experiment. Roton bound states with s-symmetry are unlikely to exist because the repulsive short range interaction is important for s states.

The long range interaction and bound states are shown to be unimportant in the scattering of two rotons. A variational calculation of the scattering of 2 rotons is made by assuming various forms for the short range repulsive interaction. Agreement with experiment is obtained if the potential is strong and has a range $1 - 2 \text{ \AA}^0$.

There are various places at which the theory could be improved. The total intensity of the scattered light is too large by a factor of 40. It would be hoped that a better calculation of the 4-particle correlation function would improve the agreement with experiment. A better calculation could be made of the 2 roton bound state by beginning with a variational function for 2 rotons of the Feynman-Cohen form. As the roton bound state is quite large $\sim 13 \text{ \AA}^0$, this may not lead to significantly different results from the present calculation.

APPENDIX A

Density Fluctuation Operator $\delta n(\mathbf{r}, t)$
of Liquid Helium II ⁽²²⁾

The Hamiltonian of the liquid is

$$H = \int (\epsilon(\rho) + \frac{\vec{v} \cdot \rho \vec{v}}{2}) d^3 r \quad (\text{A.1})$$

where $\epsilon(\rho)$ is the internal energy per unit volume and depends on the density ρ . At 0°K the system is in its ground state. At higher temperatures, for weakly excited states of the liquid, we can expand H in terms of ρ_1 and \vec{v} , where ρ_1 and \vec{v} are the density and velocity fluctuation.

$$H = \int \epsilon(\rho_0) d^3 r + \frac{1}{2} \int \rho_0 v^2 d^3 r + \frac{1}{2} \int \phi(\vec{r}_1, \vec{r}_2) \rho_1(\vec{r}_1) \rho_1(\vec{r}_2) d^3 r_1 d^3 r_2 .$$

The linear term in ρ_1 is zero because of the homogeneity of the medium. $\phi(\vec{r}_1, \vec{r}_2)$ is the second functional derivative of the energy density $\epsilon(\rho)$ and is a function of $\vec{r}_1 - \vec{r}_2$. Expand ρ_1 , \vec{v} and ϕ in Fourier series we have

$$H = E_0 + \frac{1}{2V} \int \left(\rho_0 \vec{v}_q \vec{v}_{-q} + \phi_q \rho_q \rho_{-q} \right) . \quad (\text{A.2})$$

From the linearized equation of conservation of mass we obtain, for longitudinal modes,

$$\vec{v}_q = i \dot{\rho}_q \vec{q} / \rho_0 q^2 .$$

Equation (A.2) can be written as

$$H = E_0 + \frac{1}{2V} \sum_q \left(\frac{\dot{\rho}_q \dot{\rho}_{-q}}{\rho_0 q^2} + \phi_q \rho_q \rho_{-q} \right) . \quad (\text{A.3})$$

The momentum conjugate to $\rho_{\vec{q}}$ is

$$\pi_q = \frac{\partial H}{\partial \dot{\rho}_q} = \dot{\rho}_{-q} / V \rho_0 q^2$$

and

$$[\rho_q, \pi_{q'}] = i\hbar \delta_{\vec{q}, \vec{q}'} .$$

Equation (A.3) can now be written as a sum of Hamiltonians for harmonic oscillators with frequency

$$\omega_q^2 = \rho_0 q^2 \phi_q . \quad (\text{A.4})$$

$$H = E_0 + \sum_q \hbar \omega_q \left(\frac{\rho_0 q^2 V}{2\hbar \omega_q} \pi_{\vec{q}} \pi_{-\vec{q}} + \frac{\omega_q}{2\hbar \rho_0 q^2 V} \rho_{\vec{q}} \rho_{-\vec{q}} \right) .$$

Define a creation operator $a_{\vec{q}}^+$ and an annihilation operator $a_{\vec{q}}$:

$$a_{-\vec{q}}^+ = \left(\frac{\rho_0 q^2 V}{2\hbar \omega_q} \right)^{1/2} \pi_{-\vec{q}} + i \left(\frac{\omega_q}{2\hbar \rho_0 q^2 V} \right)^{1/2} \rho_q \quad (\text{A.5})$$

$$a_{\vec{q}} = \left(\frac{\rho_0 q^2 V}{2\hbar \omega_q} \right)^{1/2} \pi_{-\vec{q}} - i \left(\frac{\omega_q}{2\hbar \rho_0 q^2 V} \right)^{1/2} \rho_q \quad (\text{A.6})$$

Then

$$H = E_0 + \sum_{\vec{q}} (a_{\vec{q}}^+ a_{\vec{q}} + \frac{1}{2}) \hbar \omega_{\vec{q}}$$

and

$$[a_{\vec{q}}, a_{\vec{q}'}^+] = \delta_{\vec{q}, \vec{q}'}$$

$$[a_{\vec{q}}, a_{\vec{q}'}] = 0 = [a_{\vec{q}}^+, a_{\vec{q}'}^+]$$

From Eqs. (A.4) and (A.5) we obtain

$$\pi_{-\vec{q}} = \left(\frac{\hbar \omega_q}{2\rho_0 q^2 V} \right)^{1/2} (a_{-\vec{q}}^+ + a_{\vec{q}}) \quad (\text{A.7})$$

$$\rho_q = \frac{1}{2i} \left(\frac{2\hbar \rho_0 q^2 V}{\omega_q} (a_{-\vec{q}}^+ - a_{\vec{q}}) \right) \quad (\text{A.8})$$

The ground state energy

$$E_g = E_o + \sum_q \hbar \omega_q / 2 .$$

Since the average kinetic energy is equal to the average potential energy, we have from Eq. (A.3)

$$\frac{\phi_q}{V} \langle \rho_q \rho_{-q} \rangle = \frac{\hbar \omega_q}{2} . \quad (\text{A.9})$$

Then from Eqs. (A.4) and (A.9) we can write

$$\hbar \omega_q = \frac{\hbar^2 q^2}{2m S(\vec{q})} \quad (\text{A.10})$$

where

$$S(q) = \frac{\langle \rho_q \rho_{-q} \rangle}{m \rho_o V} \quad (\text{A.11})$$

is the liquid structure factor. In the Heisenberg representation

$$\begin{aligned} \rho_{\vec{q}}(t) &= e^{i\hbar t/\hbar} \rho_{\vec{q}}(0) e^{-i\hbar t/\hbar} \\ &= \frac{1}{2i} \left(\frac{2\hbar \rho_o q^2 v}{\omega_q} \right)^{1/2} \left(a_{\vec{q}}^+ e^{i\omega_{-q} t} - a_{\vec{q}} e^{-i\omega_q t} \right) . \end{aligned}$$

The number density fluctuation operator is then

$$\delta n_{\mathbf{q}}(t) = \frac{1}{m} \rho_{\mathbf{q}}(t) = \frac{1}{2i} \left(\frac{2\hbar n_0 q^2 v}{m \omega_{\mathbf{q}}} \right)^{1/2} (a_{-\mathbf{q}}^+(t) - a_{\mathbf{q}}(t)) \quad (\text{A.12})$$

APPENDIX B

Evaluation of the Function $F(\vec{P}, \Omega)$

$$\vec{F}(\vec{P}, \Omega) = \sum_{\vec{p}} (\Omega - \epsilon(\vec{p}) - \epsilon(\vec{P} - \vec{p}) + i\gamma)^{-1} \quad (\text{B.1})$$

where

$$\epsilon(\vec{p}) = \Delta_0 + (|\vec{p}| - p_0)^2 / 2\mu_0 .$$

$$(1) \quad \vec{P} = 0 .$$

$$F(0, \Omega) = \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{p^2 dp}{\Omega - 2\Delta_0 - (p - p_0)^2 / \mu_0 + i\gamma}$$

where $\gamma \rightarrow 0^+$. As p_0 is much larger than $\sqrt{\mu_0(\Omega - 2\Delta_0)}$, we can make the approximation

$$F(0, \Omega) = \frac{\mu_0 p_0^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{dp'}{\mu_0 e - p'^2 + i \mu_0 \gamma}$$

with $e = \Omega - 2\Delta_0$.

$$F(0, \Omega) = \frac{-i \mu_0 p_0^2}{2\pi \sqrt{\mu_0 e}} \quad \text{if } e > 0 , \quad (\text{B.2})$$

$$F(0, \Omega) = \frac{-\mu_0 p_0^2}{2\pi \sqrt{\mu_0} b} \quad \text{if} \quad e = -b < 0 \quad . \quad (\text{B.3})$$

$$(2) \quad \sqrt{\mu_0} e > P > 0 \quad .$$

We consider

$$\begin{aligned} & \epsilon(\vec{p} + \frac{\vec{P}}{2}) + \epsilon(\vec{p} - \frac{\vec{P}}{2}) \\ &= 2\Delta_0 + \frac{1}{2\mu_0} (|\vec{p} + \frac{\vec{P}}{2}| - p_0)^2 + \frac{1}{2\mu_0} (|\vec{p} - \frac{\vec{P}}{2}| - p_0)^2 \\ &\approx 2\Delta_0 + \frac{1}{\mu_0} ((p - p_0)^2 + \frac{P^2}{4} \cos^2 \theta) \end{aligned}$$

$$\text{where} \quad \cos \theta = \frac{\vec{p} \cdot \vec{P}}{pP} \quad .$$

$$\begin{aligned} F(\vec{P}, \Omega) &= \frac{\mu_0 p_0^2}{4\pi^2} \int_{-\infty}^{\infty} \int_{-1}^1 \frac{dp' dx}{\mu_0 e - p'^2 - P^2/4 x^2 + i\gamma} \\ &= -i \frac{\mu_0 p_0^2}{4\pi} \int_{-1}^1 \frac{dx}{\sqrt{\mu_0 e - P^2/4 x^2}} \\ &= -i \frac{\mu_0 p_0^2}{\pi P} \sin^{-1} \frac{P}{2\sqrt{\mu_0} e} \quad . \end{aligned} \quad (\text{B.4})$$

$$(3) \quad p \gg P > \sqrt{\mu_0 e} .$$

$$\begin{aligned}
 F(\vec{P}, \Omega) &= -i \frac{\mu_0 p_0^2}{4\pi} \int_{-1}^1 \frac{dx}{\sqrt{\mu_0 e - P^2/4 x^2}} \\
 &= - \frac{\mu_0 p_0^2}{2\pi P} \left[\int_{-P/2}^{-\mu_0 e} \frac{dy}{\sqrt{y^2 - \mu_0 e}} + \int_{\mu_0 e}^{P/2} \frac{dy}{\sqrt{y^2 - \mu_0 e}} \right. \\
 &\quad \left. + i \int_{-\mu_0 e}^{\mu_0 e} \frac{dy}{\sqrt{\mu_0 e - y^2}} \right] \tag{B.5} \\
 &= - \frac{\mu_0 p_0^2}{2\pi P} \left[\log \frac{1 + \sqrt{1 - 4 \mu_0 e / P^2}}{1 - \sqrt{1 - 4 \mu_0 e / P^2}} + i \pi \right] .
 \end{aligned}$$

$$(4) \quad P \gg \sqrt{\mu_0 e}$$

$$F(P, \Omega) = \frac{1}{(2\pi)^2} \int_0^\infty p^2 dp \int_{-1}^1 d \cos \theta \frac{1}{\Omega - \epsilon(\vec{p}) - \epsilon(\vec{p} + \vec{P}) + i \gamma} .$$

Let us define $u = |\vec{p} + \vec{P}|$ then $udu = p P d \cos \theta$

$$\begin{aligned}
 F(\vec{P}, \Omega) &= \frac{2\mu_0}{(2\pi)^2} \int_0^\infty p^2 dp \int_{-1}^1 d\cos\theta \frac{1}{2\mu_0 e - (p - p_0)^2 - (u - p_0)^2 + i \gamma \mu_0} \\
 &= \frac{\mu_0}{2\pi^2} P \int_0^\infty p dp \int_{|p-P|}^{|p+P|} \frac{udu}{2\mu_0 e - (p-p_0)^2 - (u-p_0)^2 + i \gamma \mu_0}
 \end{aligned}$$

Let $u - p_0 = x$ and $p - p_0 = y$.

$$F(\vec{P}, \Omega) = \frac{\mu_0 p_0^2}{2\pi^2 P} \int_{-p_0}^\infty dy \int_{|y-P+p_0|-p_0}^{|y+p_0+P|-p_0} dx \frac{1}{2\mu_0 e - x^2 - y^2 + i \gamma \mu_0}$$

Except for $P \approx 2p_0$, we can write

$$\begin{aligned}
 F(\vec{P}, \Omega) &\approx \frac{\mu_0 p_0^2}{2\pi^2 P} \int_{-\infty}^\infty dy \int_{|P-p_0|-p_0}^P dx \frac{1}{2\mu_0 e - (x^2 + y)^2 + i \gamma \mu_0} \\
 &= -i \frac{\mu_0 p_0^2}{2\pi P}
 \end{aligned}
 \tag{B.6}$$

APPENDIX C

Evaluation of Equation (6.27)

$$G_2 = \pi \int_{\mathbf{q}} v^2(\mathbf{q}) \delta(\Omega - \epsilon(\vec{p}_1 + \vec{q}) - \epsilon(\vec{p}_2 - \vec{q})) .$$

Since $p_1 \approx p_0 \approx p_2 \gg q$, we can expand $\epsilon(\vec{p}_1 + \vec{q})$ and $\epsilon(\vec{p}_2 - \vec{q})$ in terms of q/p_0 .

$$\begin{aligned} \epsilon(\vec{p}_1 + \vec{q}) + \epsilon(\vec{p}_2 - \vec{q}) &= 2\Delta_0 + \frac{(|\vec{p}_1 + \vec{q}| - p_0)^2}{2\mu_0} \\ &\quad + \frac{(|p_2 - q|^2 - p_0^2)}{2\mu_0} \\ &\approx 2\Delta_0 + \frac{1}{2\mu_0 p_0^2} [(\vec{p}_1 \cdot \vec{q})^2 + (\vec{p}_2 \cdot \vec{q})^2] . \end{aligned}$$

Let \vec{p}_1 and \vec{p}_2 be on the xy-plane and $\vec{p} = (\vec{p}_1 - \vec{p}_2)/2$ and $\vec{P} = \vec{p}_1 + \vec{p}_2$. Then

$$\delta(\Omega - \epsilon(\vec{q} + \vec{p}_1) - \epsilon(\vec{p}_2 - \vec{q})) = \delta(\Omega - 2\Delta_0 - \frac{1}{2\mu_0 p_0^2} (\vec{p} \cdot \vec{q}) + (\frac{\vec{P}}{2} \cdot \vec{q})^2)$$

and

$$\begin{aligned}
 G_2 &= \frac{\pi}{(2\pi)^3} \int_{-\infty}^{\infty} dq_z \int_{-\infty}^{\infty} dq_x dq_y v^2(q) \delta(\Omega - 2\Delta_0 - \frac{(\vec{p} \cdot \vec{q}) + (\vec{P}/2 \cdot \vec{q})^2}{\mu_0 P_0^2}) \\
 &= \frac{\mu_0 P_0^2}{4\pi p P} \int_{-\infty}^{\infty} dq_z v^2(\sqrt{\mu_0(\Omega - 2\Delta_0) + q_z^2}) .
 \end{aligned}$$

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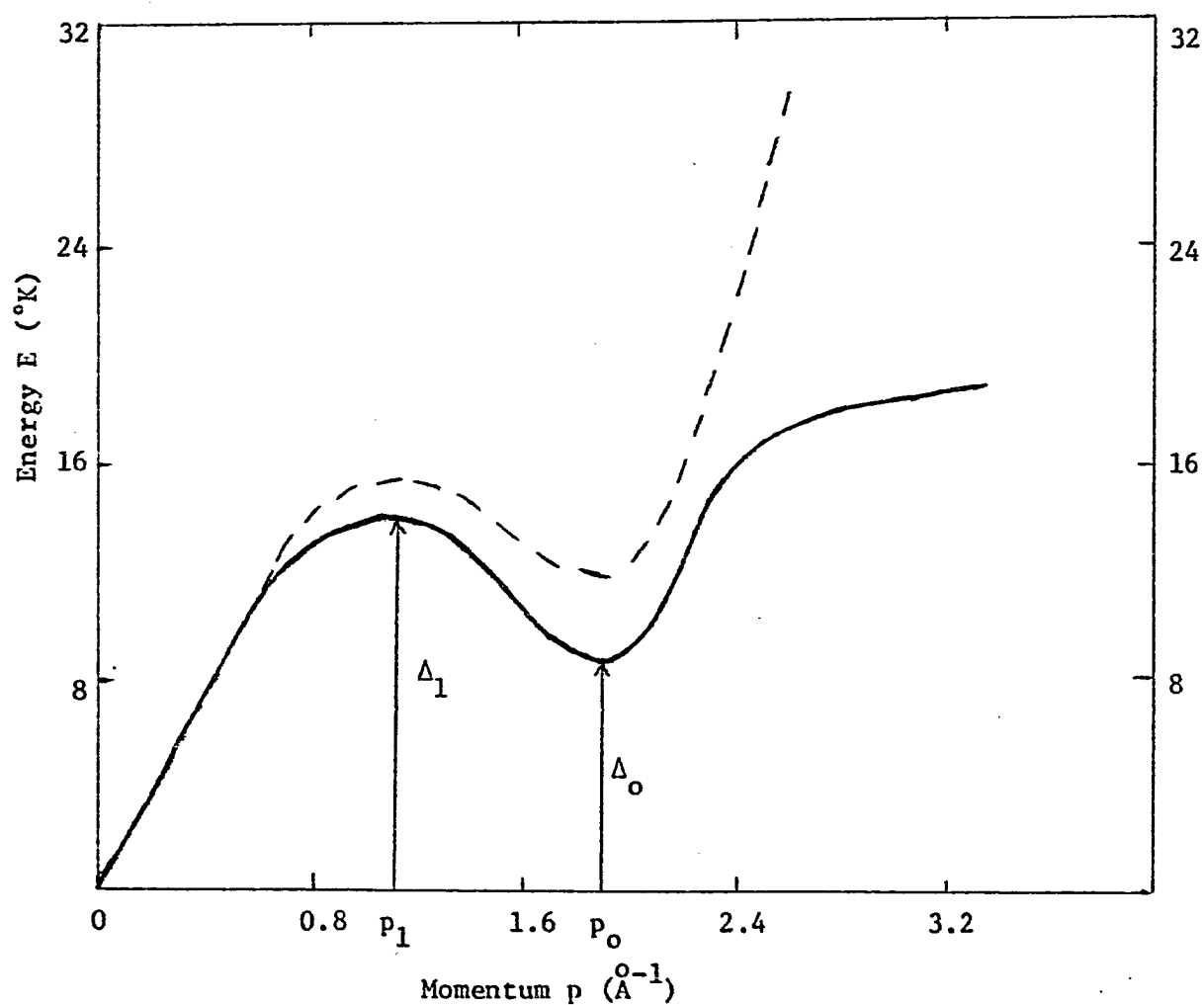


Figure 1 The excitation spectrum obtained from neutron scattering experiment (solid line) and the excitation spectrum as derived by Feynman and Cohen (dashed line).

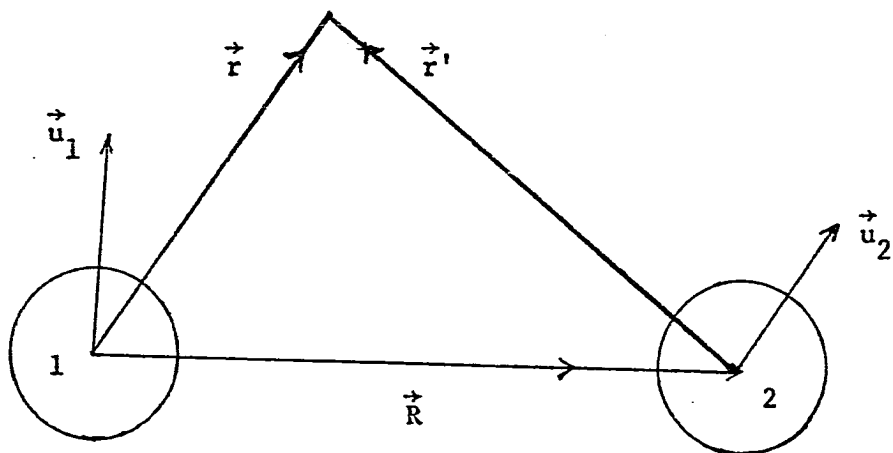


Figure 2 Two spheres 1 and 2 moving with velocities \vec{u}_1 and \vec{u}_2 in an ideal fluid. \vec{r} and \vec{r}' are spherical coordinates with origins at the centers of spheres 1 and 2, respectively.